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WAVE-PARTICLE INTERACTIONS IN THE MAGNETOSPHERIC PLASMA

Michael Schulz

Battelle-Pacific Northwest Laboratories

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ATR-74(7420)-1

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IN THE MAGNETOSPHERIC PLASMA

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BATTELLE PACIFIC-NORTHWEST LABORATORIES

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FOREWORD

This collection of technical papers, some of which are intended for journal publication, comprise the final report on the magnetospheric modification study carried out at The Aerospace Corporation during 1973 and 1974. This work was undertaken following prior research on theoretical aspects of artificial plasma injection into the magnetosphere. The objective was to provide further support of a theoretical nature for magnetospheric modification programs.

Three specific topics were mentioned in the work statement of the present research. The objectives were to study the effects of fast electrons in bounce resonance with amplifying ULF waves, to study the modification of ULF and VLF growth rates by gradient and trapping effects in an inhomogeneous plasma, and to study the excitation of ULF and VLF signals within a plasma cloud by means of a modulated ion beam. Significant (but uneven) progress was made on each of these topics during the period of research. Reports of progress made on related topics, not specifically mentioned in the work statement, are also included here.

The present collection of papers includes contributions to both the formal and the heuristic theory of wave-particle interactions in the magnetospheric plasma. The first and eighth papers pay special attention to the formulation based on the adiabatic invariants of charged-particle motion, while the sixth and seventh papers concentrate on the effective quantization of wave spectra. The second and third papers provide a careful treatment of some uniform-plasma instabilities and their geophysical consequences, while the fifth offers a more heuristic view

of the wave-particle interaction. The fourth paper provides a precise evaluation of particle lifetimes in the strong-diffusion limit, which plays an important role in both the formal and the heuristic aspects of wave-particle theory. The last three papers (9-11) relate to magnetospheric physics as an experimental science, in particular to the use of particle beams as radiators and amplifiers of wave energy.

Certain contributions, notably the eighth and ninth, represent only rudimentary remarks on the problems in question. However, the first seven papers present research results in essentially finished form. Relatively few refinements on those seven topics are planned prior to publication.

CRUDE APPROXIMATIONS TO SOME ASPECTS OF THREE-DIMENSIONAL MAGNETOSPHERIC DYNAMICS

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1. INTRODUCTION

It is quite well-understood in principle how to formulate and solve dynamical problems in a three-dimensional magnetosphere (i.e., one in which all three adiabatic invariants M, J, Φ come into play), given the relevant diffusion coefficients, loss rates, and so forth (Haerendel, 1968; Lanzerotti and Schulz, 1973). But there are very severe practical difficulties in carrying out a truly three-dimensional calculation, and practically none exist in the literature. Two-dimensional calculations abound: radial diffusion of equatorially-mirroring ($J = 0$) particles, pitch-angle scattering at fixed L (or fixed Φ). In many cases, this is not good enough; for example, most data on energetic trapped alpha particles (e.g., Fennell et al., 1973) is off-equatorial, but theory (e.g., Cornwall, 1972) has concentrated on $J = 0$ particles.

The purpose of the present work is to go one small step beyond purely qualitative discussions of three-dimensional problems by providing a crude,

semi-quantitative overview of the essential features of such problems. This work is not intended in any way to replace real three-dimensional calculations, but it should provide space physicists with a road map for following the results of such calculations, should they be carried out. Our main emphasis is on the change of pitch-angle anisotropy due to radial diffusion, with or without pitch-angle diffusion.

In Section 2, an approximate formula for the variation of equatorial pitch angle with L during M- and J-conserving transport processes is given. The virtue of this approximation is that it yields a simple, immediately interpretable scaling law for distribution functions which are power laws in energy and pitch angle, when the transport processes are independent of M and J. Section 3 discusses a simple problem where M- and J-dependent transport processes are involved, and Section 4 discusses an approximation to the problem of coupled radial diffusion-pitch-angle instability. Here radial diffusion tries to increase the pitch-angle anisotropy, and pitch-angle diffusion tries to decrease it.

2. APPROXIMATE KINEMATICS

The theoretician formulates theories in terms of the adiabatic invariants M, J, and Φ, while experimentalists measure energy E, equatorial pitch-angle α, and L. In a dipole field, we may (and do) choose $\Phi = L^{-1}$, but the relationship between L, M and J, are on the one hand, and E and $y \equiv \sin\alpha$ on the other, is usually given numerically (e.g., Nakada, Dungey, and Hess, 1965). For non-relativistic motion, y is a function of L only at fixed M and J. In the notation of Lanzerotti and Schulz (1973) the first invariant is

$$M = \frac{p_\perp^2}{2mB} = \frac{Ey^2L^3}{2B_o} \tag{2-1}$$

and the second invariant is

$$J = \oint p_{||}\, ds = 2pLa\, Y(y) \tag{2-2}$$

where $Y(y)$ is a complicated function of pitch-angle. Here $p_\perp$, $p_{||}$, p are momentum variables, m is mass, $E = p^2/2m$, $B_o = .31$ gauss, a = one Earth radius. It follows upon elimination of p that

$$\frac{Y(y)}{L^{\frac{1}{2}}y} = \text{const. at fixed } M, J \tag{2-3}$$

It is not hard to see that (2-3) can be written in the equivalent form

$$y^{h(L,y_o)} = \text{const.} = y_o^{h(L_o,y_o)} \tag{2-4}$$

where the exponent h, depending on both L and y_o, can be expressed in terms of the function Y/y and its inverse. (The reference value y_o depends on L_o, and on J^2/M, from (2-1) and (2-2).) Equation (2-4) would be much

more useful if the dependence of h on y_o could be suppressed, at least approximately, for then an interesting scaling law holds for the adiabatic transformation of a distribution function which is the product of a power low in energy and in y. A numerical study shows that for a wide range of values of y_o, the exponent h is indeed roughly independent of y_o. The expression

$$y^{h(L)} = y_o^{h(L_o)} \tag{2-5}$$

is valid to within 3% or better in the range $0.2 \leq y_o \leq 0.8$ (roughly 10°-60° in equatorial pitch angle) with $L_o = 7$, and $2 \leq L \leq 7$. The function $h(L)/h(L_o)$ is shown in Fig. 1. For larger or smaller values of y_o, the approximation that h is independent of y_o becomes progressively worse, but even for $0.8 \leq y_o \leq 1$, (2-5) is usefully accurate. The reason is that if y_o is sufficiently close to 1, y_o^h depends insensitively on h; thus (2-5) correctly predicts that $y = 1$ if $y_o = 1$, no matter what h is.

With the approximation that h is independent of y_o, we can extend the fundamental energy scaling laws of Nakada, Dungey, and Hess (1965) to pitch-angle scaling laws. Let a particle, initially at L_o, have energy E_o and pitch-angle variable y_o there. The conservation of M as expressed in the second form of (2-1), plus the approximate formula (2-5), yields

$$E = E_o(L_o/L)^3 \, y^{2(h/h_o)-2} \tag{2-6}$$

where E and y are the transformed values after the particle has undergone adiabatic transport (conserving M and J), and $h_o \equiv h(L_o)$. Equation (2-6) expresses the well-known result that particles with small pitch angles gain less energy than those with large pitch angles.

Let a group of particles have an initial momentum-space distribution function at $L = L_o$ of the type

$$f_o = E_o^{-a} y_o^{b} \tag{2-7}$$

and let these particles be acted upon by processes which (1) preserve M and J; (2) have transport coefficients independent of M and J. Then, according to the single-particle laws (2-5) and (2-6), the distribution function maps into a similar form

$$f_o \to f \sim E^{-a} y^{-\beta} \tag{2-8}$$

(leaving out a coefficient depending on L only) where

$$\beta(L) = -2a + (b + 2a)(h/h_o) \tag{2-9}$$

Because h/h_o increases with decreasing L, the anisotropy β increases, as is well-known, but what may not be so well-known is that most of the increase in anisotropy comes from the decreasing power law in energy. For example, the values $a = 3$, $b = 1$ roughly characterize energetic (> 50 keV) ring-current protons at $L = 7$. At $L = 4$, (2-9) yields $\beta = 2$ and at $L = 2$, $\beta = 4$. However, for the less-energetic protons at the peak (10-20 keV) of the ring current distribution at $L = 7$, $a \approx 0$, and $\beta \approx 1.1$ at $L = 4$, 1.4 at $L = 2$. Thus processes of the sort considered in this paragraph do not lead to much increase of anisotropy of the particles at the flux peak, which may very well be significant for the dynamical role of instabilities driven by pitch-angle anisotropy.

In fact, no known transport processes are independent of M and J, so the single-particle laws (2-5), (2-6) cannot be promoted to a distribution-function law such as (2-8). Moreover, no distribution function is really of

the factorizable power-law type in (2-7). Nonetheless, (2-8) and (2-9) should be a useful and rapid way of characterizing the zeroth-order change in the distribution function. In Sections 3 and 4, we go beyond the simple rule (2-9) to discuss more-or-less realistic dynamical processes, and interpret a and β in terms of suitable moments of the distribution function.

3. QUASI-REALISTIC RADIAL DIFFUSION DYNAMICS

Let f be the phase-space distribution function averaged over cyclotron phase, bounce phase, and longitude or what amounts to the same thing, the distribution function in M, J, and Φ . Suppose that f is subject to radial diffusion and a loss process, as described by:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \Phi} \left(D_{\Phi\Phi} \frac{\partial f}{\partial \Phi} \right) - \lambda f \tag{3-1}$$

(With $\Phi = L^{-1}$, $D_{\Phi\Phi} = L^{-4} D_{LL}$.) Here the diffusion itself is one-dimensional, but if λ and D_{LL} depend on M and J , one has a non-trivial complication of the sort discussed in the last Section, in converting from M and J to E and y .

Let us parametrize D_{LL} and λ by power laws in L , E_o , and y_o where E_o and y_o (equivalent to M and J) are the energy and pitch-angle variable at a reference L-value L_o (we choose $L_o = 7$ in what follows):

$$D_{LL} = D_o L^{\alpha} E_o{}^{s} y_o{}^{t} , \quad \lambda = \lambda_o L^{-\beta} E_o{}^{p} y_o{}^{q} \tag{3-2}$$

As Haerendel (1968) has indicated, the time-stationary solution of (3-1) is a linear combination of the functions

$$L^{-m/2} K_{\nu}(z) , \quad L^{-m/2} I_{\nu}(z) \tag{3-3}$$

with

$$m = \alpha - 3, \ n = \alpha + \beta - 2, \ \nu = -m/n,$$

$$z^2 = \left(\frac{4\lambda_o}{n^2 D_o} \right) L^{-n} E_o{}^{p-s} y_o{}^{q-t} \tag{3-4}$$

Here K_ν, I_ν are the usual modified Bessel functions. To express the solution in terms of E and y, one uses the relations (2-5) and (2-6).

There are two main processes for radial diffusion: magnetic impulses, and electrostatic fluctuations. For both, we take the usual value $\alpha = 10$. With a fluctuation power spectrum falling like ω^{-2} (for both electric and magnetic variations), magnetic diffusion has $s \approx 0$, $t \approx 2.7$, while for electrostatic diffusion, $s \approx -2$, $t \approx 0$. Less is known about the dominant loss processes; for simplicity, we take $p = q = \beta = 0$, hence $n = 8$, $\nu = -7/8$. Other cases are easily worked out. The positive power t for magnetic diffusion reflects the well-known fact that magnetic diffusion is weak at high latitudes, i.e., small pitch angles.

First, we discuss electrostatic diffusion. For not-too-relativistic electrons and for protons, Cornwall (1972) has estimated that for electrostatic fluctuations $D_o \sim 10^{-4}$ (keV)^2/day. Ring-current protons have an effective lifetime of a day or so, thus $\lambda_o^{-1} \sim 1$ day. In this case, (3-4) gives $z \sim 25\, L^{-4} E_o$ with E_o in keV. The decrease of z with increasing L indicates that the dynamics are diffusion-dominated at large L $(z \ll 1)$, loss-dominated at small L $(z \gg 1)$. The boundary condition $f \simeq 0$ at $L \simeq 1$, where $z \gg 1$, requires us to use only the K_ν solution, with the asymptotic behavior

$$K_\nu(z) \sim (\pi/2z)^{\frac{1}{2}}\, e^{-z}\,, \quad z \gg 1 \tag{3-5}$$

The full solution, satisfying the appropriate boundary condition at $L = L_o$, is:

$$f(L, E_o, y_o) = \left(\frac{L_o}{L}\right)^{m/2} f\,(L_o, E_o, y_o)\, K_\nu(z)/K_\nu(z_o) \tag{3-6}$$

where $z_o = z(L = L_o)$.

In the diffusion-dominated regime $K_\nu(z)/K_\nu(z_o) \simeq 1$, and the remarks of the last Section hold: power-law distribution functions map into power laws, according to (2-9). However, in the loss-dominated regime there are new effects: power-law scaling breaks down, but the local pitch-angle anisotropy at fixed energy (defined in (3-10)) increases over the value given in (2-9). The reason for the increase is that particles with high energy diffuse electrostatically more slowly than low-energy particles, so that the energy spectrum becomes steeper; that is, in effect the parameter a of (2-9) is not constant, but increasing with decreasing L .

With the aid of (3-5), the loss-dominated solution is (aside from an overall multiplicative function of L)

$$f \sim E_o^{-\frac{1}{2}} e^{z_o - z} f(L_o, E_o, y_o) \tag{3-7}$$

Take the initial distribution to be of the power-law type (2-7), and apply the transformation laws (2-5) and (2-6) to find

$$f \sim E^{a-\frac{1}{2}} y^{\beta+r} \exp\left(- .07 \left(\frac{E}{L} \right) y^{-2r} \right) \tag{3-8}$$

where β is given in (2-9), and

$$r = (h/h_o) - 1 \tag{3-9}$$

Define the local pitch-angle anisotropy as

$$\gamma \equiv (y/f)\ \partial f/\partial y \tag{3-10}$$

This definition has physical significance; the growth rate of the electromagnetic cyclotron instability is essentially an integral over energy of $f\gamma$ as given in (3-10). For a power-law distribution, γ is just the power of y .

For the distribution function (3-8), it is easy to find

$$\gamma(E,y) = \beta + r + .14r(E/L)y^{-2r} \tag{3-11}$$

The anisotropy has significantly increased over the value β derived from the scaling law (2-9). At $L = 3,5$, where from Figure 1 $r = 0.2$, $\gamma \simeq \beta + 0.2 + 9 \times 10^{-3}E$ and a 100-keV particle has $\gamma \simeq \beta + 1.1$; at $L = 2$, $\gamma \simeq \beta + 2.2$. This rapid increase in anisotropy would represent a significant increase in the free energy available to drive instabilities, were it not for the fact that it is the loss mechanism which is responsible for the anisotropy increase; the available free energy may be increased or decreased as a result of the process discussed here.

The increase of anisotropy can be directly traced to the fact that z decreases with increasing y, at fixed L and E. Thus the same phenomenon occurs for magnetic diffusion. The general condition for the anisotropy to increase above β with decreasing L is

$$2r(p - s) + (t - q)(r + 1) > 0 \tag{3-12}$$

which is satisfied for magnetic diffusion and constant λ for $t > 0$. Using a small magnetic diffusion coefficient, such as $D \sim 10^{-10}L^{10}$, leads to tremendous increases in anisotropy, but far more tremendous decreases in total flux. An interesting balance between anisotropy increase and flux decrease can only be achieved when z is not much larger than, or much smaller than, one, that is, on the boundary of loss-dominated and diffusion-dominated transport.

It is worth noticing that this sort of process may actually decrease the anisotropy (compared to that given in (2-9)) for low-energy protons. Here charge-exchange losses are important, and for them p is negative which works the wrong way for condition (3-12) to be satisfied. Moreover, for very low-energy

protons ($\lesssim$ 10 keV at $L = 4$) Cornwall (1972) has estimated that $s \simeq 0$, $\alpha = 6$. Again, this behavior of the diffusion coefficient works the wrong way in (3-12).

4. COUPLED RADIAL DIFFUSION AND PITCH-ANGLE DIFFUSION

In the presence of transport processes which violate M and J as well as Φ, equation (3-1) is extended to (the loss term λ is dropped, for the sake of brevity):

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \Phi}\left(D_{\Phi\Phi} \frac{\partial f}{\partial \Phi} \right) + Q^{-1} \sum_{ij} \frac{\partial}{\partial x^i}\left(Q D^{ij} \frac{\partial f}{\partial x^j} \right) \tag{4-1}$$

where the variables x^1, x^2 are equivalent to M and J, and

$$Q = \frac{\partial(M,J)}{\partial(x^1,x^2)} \tag{4-2}$$

The presence of the Jacobian Q is demanded by the canonical nature of the variables M, J, Φ. It is convenient to choose the variables x^i to be the velocity components at the equator:

$$(x^1,x^2) = (v_\perp, v_{||}) \; ; \quad v_\perp = \left(\frac{2E}{m} \right)^{\frac{1}{2}} y, \; v_{||} = \left(\frac{2E}{m} \right)^{\frac{1}{2}} (1-y^2)^{\frac{1}{2}} \tag{4-3}$$

Then one readily finds that

$$Q \sim v_\perp (1-y^2)^{\frac{1}{2}} \, T(y) \tag{4-4}$$

(a factor independent of M and J is omitted), where $T(y)$ is the normalized bounce time:

$$T_B = (4mLa/p) \, T(y) \tag{4-5}$$

The diffusion coefficients D^{ij} are averaged over bounce phase, cyclotron phase, and longitude; thus they differ from the usual locally defined diffusion coefficients of quasi-linear theory. For electromagnetic cyclotron waves

propagating parallel to the static field, the local equivalent of the second term on the right of (4-1) is in the resonant approximation (Kennel and Engelmann, 1966)

$$\sum_k v_\perp^{-1} G_k (v_\perp D_k G_k f) \tag{4-6}$$

$$G_k = \left(\frac{\omega}{k} - v_{||} \right) \frac{\partial}{\partial v_\perp} + v_\perp \frac{\partial}{\partial v_{||}} \tag{4-7}$$

$$D_k = \frac{\pi}{2} \left| \frac{eb_k}{mc} \right|^2 \delta(kv_{||} - \omega_k + \Omega) \tag{4-8}$$

In these equations, $v_\perp$ and $v_{||}$ refer to local components, not components of the equatorial velocity. By comparison with (4-1), it is reasonable to assume that the bounce-phase averaging process yields equations like (4-6) and (4-7), except that the explicit $v_\perp$ in (4-6) is replaced by Q (see (4-4)), and $v_\perp$, $v_{||}$ are taken to mean equatorial components, as in (4-3). Furthermore, D_k must be defined as an average over the actual wave fields, so the form (4-8) is not really appropriate. As the reader will soon see, the use of (4-8) does not invalidate the formulas below.

The quasi-linear description is completed with the equation for wave energy:

$$\frac{\partial W_k}{\partial t} + \underset{\sim}{\nabla} \cdot (\underset{\sim}{V}_G W_k) = 2\gamma_k W_k \tag{4-9}$$

where $\underset{\sim}{V}_G$ is the group velocity, γ_k the growth rate, and the wave energy W_k is given by

$$W_k = \frac{|b_k|^2}{8\pi} \left[1 + n^{-2} \frac{\partial}{\partial \omega} (\omega n^2) \right] = \frac{|b_k|^2}{8\pi} \frac{2v_{ph}}{V_G} \tag{4-10}$$

Here n is the index of refraction, and v_{ph} the phase velocity. Again, these equations must be interpreted as suitable averages over the bounce phase and longitude, which amounts to replacing $\tilde{\nabla}\cdot(\tilde{V}_G W_k)$ by $-(V_G/\ell)W_k$, where $\ell \sim La$ is a length characteristic of the field line.

The expression for the growth rate γ_k must be such that it correctly accounts for the transfer of energy between waves and particles. As with all the other formulas here, a spatial average is carried out, so γ_k is not given by the usual local expression. The only difference is the appearance of the Jacobian Q :

$$\gamma_k = \pi^2 V_G \frac{e^2}{mc^2} \int d^3v\, Q\, \delta(kv_{||} - \omega_k + \Omega) G_k\, f \qquad (4\text{-}11)$$

Here, of course, the multiplicative constant omitted in the definition (4-4) of Q matters. Since (4-11) gives the correct local growth rate if Q is set equal to $v_\perp$ with the usual velocity-space distribution function, and since $(1-y^2)^{\frac{1}{2}}\, T(y)$ is of $O(1)$ if y is not too close to one, this multiplicative constant is itself nearly one. The final value of this constant should be chosen on phenomenological grounds (which go beyond the scope of this paper) having to do with the bounce-phase averaging, loss of resonance in the inhomogeneous magnetic field of the earth, etc. For illustrative purposes, we take this constant to be exactly one.

Equations (4-1) - (4-11) are a truly formidable set of non-linear partial differential equations in four variables, with two unknowns. It is to be hoped that someone will tackle these equations in their full complexity some day with the help of computers, but even if this is done, the results will be as difficult to interpret simply as if they were experimental data. It appears useful to extract from these equations a simpler, approximate set of new equations, which are both easier to interpret and easier to compute. In view

of the fairly simple results of Sections 2 and 3, such a simplified set of equations might make reference to an effective power-law in energy and in pitch-angle. Equivalently, one may take velocity-space moments of the quasi-linear particle equation (4-1)(e.g., Roux and Solomon, 1971; Hamasaki and Krall, 1973). Three such equations suffice: for the number density, the density of total energy, and the density of perpendicular energy.

Because the derivatives with respect to Φ (or L) in (4-1) are to be taken at fixed M and J, it is necessary at any given L to express the distribution function f in terms of M, J, and L, and then to integrate over M and J. It is not hard to show that

$$dMdJ = 2\pi C L^4 Q \, dv_\perp dv_{||} \tag{4-12}$$

where $Q = v_\perp (1-y^2)^{\frac{1}{2}} T(y)$, and C is a universal constant, independent of M, J, and L. By appropriate choice of units for M and J, we may choose $C = 1$. Define

$$N(L) = 2\pi \int Q \, dv_\perp dv_{||} \, f = L^{-4} \int dMdJ \, f \tag{4-13}$$

Since $\int dMdJ \, f$ is (proportional to) the total number of particles per unit Φ, $L^2 N(L) dL$ is the total number of particles between two L-shells separated by dL, modulo a universal constant. Similarly we define moments of any function $G(M,J)$:

$$\langle NG \rangle = 2\pi \int Q \, dv_\perp dv_{||} G \, f \tag{4-14}$$

If D_{LL} in (4-1) depends on M and J, the procedure of taking moments of (4-1) yields more unknowns than there are equations. The only way around this drawback is to assume a functional form for the distribution function f ;

in order to make contact with Sections 2 and 3, we assume a double power-law as in (2-8), where both powers may depend on L :

$$f(L,E,y) = F(L)E^{-a}y^{b} \tag{4-15}$$

Now given a particular form of D_{LL} (e.g., (3-2)), all moments are expressed in terms of the three quantities F,a,b as well as the lower cut-off energy at which the energy integration is terminated. To express f in terms of L,M,J instead of L,E,y, the adiabatic relations (2-5) and (2-6) are used. It is a simple matter to multiply (4-1) by 1, E, and $E_\perp$ and integrate to come to the three moment equations

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial \Phi} < D_{\Phi\Phi} \frac{\partial N}{\partial \Phi} > \tag{4-16}$$

$$\frac{\partial <NE>}{\partial t} = \frac{\partial}{\partial \Phi} < D_{\Phi\Phi}\, E \frac{\partial N}{\partial \Phi} > - \sum_k 2\gamma_k W_k \tag{4-17}$$

$$\frac{\partial}{\partial t} <NE_\perp> = \frac{\partial}{\partial \Phi} < D_{\Phi\Phi}\, E_\perp \frac{\partial N}{\partial \Phi} > - \sum_k 2\gamma_k \left(\frac{\Omega}{\omega_k} \right) W_k \tag{4-18}$$

In deriving (4-17) and (4-18), (4-6) to (4-8), (4-9) and (4-10) were used. In the absence of energy transport by radial diffusion or convective loss of waves, (4-17) plus (4-9) would express conservation of total energy $<NE> + \sum W_k$; it is for this reason that the expression (4-11) for γ_k is used.

The set of equations must be completed with some reference to the wave equation (4-9). The shortest time scale in the problem corresponds to the inverse of the maximum growth rate (the maximum being taken as frequency ω_k or wave number k is varied), which we denote $\bar{\gamma}$. Of course, $\bar{\gamma}$ no longer depends on k, but only on such quantities as a, b, N, etc. Because the wave growth time is short compared to the radial diffusion time, (4-9) can be

replaced by the simpler equation

$$\bar{\gamma} = \bar{V}_g / \ell \tag{4-19}$$

where $\bar{V}_g$ is the group velocity for the wave number of maximum growth, and ℓ a length characteristic of the field line. Equation (4-19) is an algebraic constraint on a, b, and F ; added to the three equations (4-16) - (4-18), one has a set of four equations for four unknowns a, b, F, $\bar{W}$ (where $\bar{\gamma}\,\bar{W} = \sum \gamma_k W_k$). In the time-stationary case ($\partial/\partial t = 0$), these equations are non-linear ordinary differential equations in the single variable L , thus considerably simpler than the original non-linear partial differential equations (4-1), (4-9), and (4-11). It is even possible to make some analytic progress with these simpler equations, but it would be premature to report on this work now.

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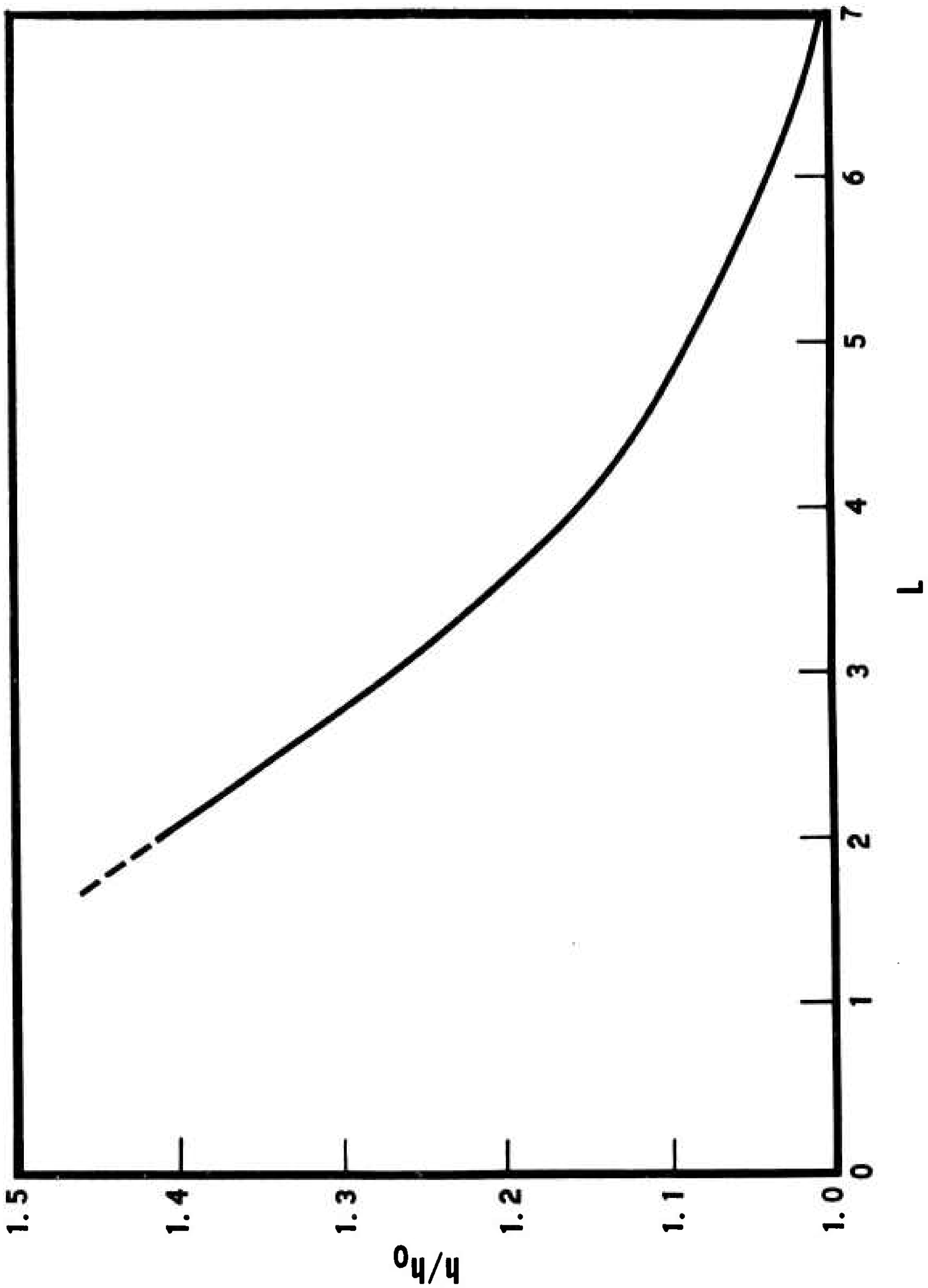
Roux, A., and J. Solomon: 1971, J. Atmos. Terrestr. Phys. 33, 1457.

FIGURE CAPTION

Variation of the pitch-angle exponent with L .


1.5
1.4
1.3
1.2
1.1
1.0
h/h$_0$
0
1
2
3
4
5
6
7
L

EFFECT OF INSTABILITIES ON THE RING CURRENT

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ABSTRACT

After years of investigation, the role of instabilities (notably in electromagnetic cyclotron modes) in magnetospheric ring-current dynamics is not yet settled. In this paper, experimental evidence for the effects of these and possibly other instabilities is reviewed. It is very interesting that at the present time only a small fraction of the total number of protons known to be in the storm-time ring current is observed to be directly precipitated at low altitudes, in part because of inadequate experimental coverage, and possibly in part because these protons lose much of their energy before precipitating and are thus not easily seen in most detectors. There is strong evidence that ring-current dynamics occurs off the equator outside the plasmapause; one possible mechanism is an electrostatic instability, which is probably quenched inside the plasmapause.

1. INTRODUCTION

If the author had written a paper with this title a couple of years ago, he would have been able to present a reasonably consistent theoretical picture of ring-current dynamics, with the ion electromagnetic cyclotron (EMC) instability, occurring just inside the plasmapause, being responsible [Cornwall et al., 1970, 1971a] for the observed termination of the ring current just inside the plasmasphere [Russell and Thorne, 1970] and for the occurrence of mid-latitude stable auroral red (SAR) arcs at the same place [Chappell et al., 1971]. However, new experimental evidence acquired recently presents a mixed picture, some of it [Berg and Søraas, 1972; Williams et al., 1973] consistent with the dynamical role of EMC instabilities, and some of it [Amundsen et al., 1972; Winningham, 1972; Kleckner and Hoch, 1973; Mizera, 1973; Shelley et al., 1972] being rather difficult to interpret within the EMC instability framework. Additionally, it is a continuing embarrassment to theoreticians (or to experimentalists, depending on your point of view) that the requisite EMC-mode emissions have not been observed in the ULF frequency range near the plasmapause at storm time. This is only an embarrassment, not a fatal objection, because of the lack of sufficiently sensitive experiments, and because of the theoretical prediction [Cornwall et al., 1971a] that these waves should be absorbed by ambient electrons nearly as soon as they are generated.

The potentially damaging evidence comes solely from measurements of low-altitude precipitation, and this in itself is a very important clue. Perhaps the cause of the precipitation is located off the equator, while the EMC dynamics take place at the equator. We suggest such a mechanism in Section 3; if it, or something similar does happen, we can explain the observed pattern of proton precipitation.

Shelley et al. [1972] have seen energetic (~ 1-10 keV) atomic oxygen ions at low altitudes both inside and outside the plasmapause during storms. We have no explanation of this very puzzling set of observations. It could be that O^+ ions can absorb energy from the ring current through a mechanism involving some sort of instability, but no mechanism known to the author can explain all the observed features, especially the fact that O^+ ions are often seen considerably equatorward of the precipitating protons. We shall remain silent on this subject for the rest of the paper.

2. OBSERVATIONS

Two years ago, some observations of precipitating protons from the low-altitude polar orbiter OV1-15 were reported [Cornwall et al., 1971b]. At the time, these data were interpreted as showing precipitation just inside the plasmapause, consistent with EMC-mode dynamics. However, some more recent evidence throws doubt on this interpretation. Mizera [1973] has continued his studies of protons > 12 keV from the polar orbiters OV1-17 and OV1-19; a typical data sample is shown in Fig. 1. Note that Mizera judges the midnight plasmapause to be just equatorward of the precipitation, and it is seen that the lower the energy, the more nearly poleward the precipitation peak; both these features are precisely the opposite of the EMC instability predictions. Although not shown in the figures, the pitch-angle distributions tend toward isotropy at the peak, with a mirroring distribution equatorward of the peak; the same feature was observed on OV1-15, and on ESRO IB [Amundsen et al., 1972], also a low-altitude polar orbiter.

The energy spectra in the low-altitude loss cone are rather similar to those of equatorially trapped protons, except that they are about a factor of 10 lower (Fig. 2). Shown are storm-time distributions from Williams et al. [1973], and quiet-time distributions from Pizzella and Frank [1971]. The fact that total precipitated fluxes are only 10% of trapped fluxes rules out an isotropic equatorial distribution; isotropy is confined to the neighborhood of the equatorial loss cone, as shown schematically in Fig. 3. Mizera estimates that less than 1% of the total storm-time ring-current energy is dissipated as precipitation of protons with E > 12 keV, although it would seem, on the basis of precipitated-to-trapped flux ratio, that as much as 10% of the total energy could be precipitated. (In this regard, note that the equatorial storm-time

flux depicted in Fig. 2 was observed two years after the OV1-17 and OV1-19 data were taken.)

Equatorward of the precipitation peak, the low-altitude pitch angle distribution assumes a conventional mirroring form, which is very consistent, for protons of ~ 300 keV, with equatorially measured distributions at L ~ 2.6 as seen by OV1-14 [Fennell et al., 1973]. The comparison is shown in Fig. 4, where OV1-19 data is normalized to OV1-14 data (the absolute values were within 30% even though the measurements were taken a year apart). Thus at lower L values (presumably inside the plasmasphere) the equatorial pitch-angle distribution is nearly empty in the loss cone.

Generally similar phenomena are reported by Winningham [1972] for low-altitude protons in the 0-15 keV energy range: isotropic fluxes poleward of the plasmapause, mirror distributions inside the plasmasphere. Only a few percent of the total trapped ring-current energy is precipitated as protons of less than 15 keV.

It is extremely unlikely that any of this precipitation can be explained by the EMC instability, which requires relatively high cold-plasma densities. Fig. 5 shows the observed proton energy at the precipitation peak vs. L, compared to the estimated energy threshold (which varies inversely with cold plasma density) of Thorne and Kennel [1971], based on the EMC instability. Clearly, some other mechanism is at work; possibly the electrostatic instability discussed in the next section.

Kleckner and Hoch [1973] have observed that H arcs tend to occur simultaneously with SAR arcs, separated from the latter by about 1.6 L units (Fig. 6), independent of the SAR arc location. Mizera observes that the average 12-50 keV proton precipitation peak is about 1.3 L units poleward of the plasmapause. It is natural to suppose that the precipitation causes the H arcs,

and that the precipitation peak marks the L-value of the trapped ring-current peak. This suggestion is further supported by the observation that the H arcs can be driven by about 10% of the total available ring-current energy, consistent with the 10% precipitation level observed by Mizera.

The precipitation data strongly suggests the necessity of a mechanism which can produce an equatorial pitch-angle distribution as shown in Fig. 3. There are two possibilities: (1) a mechanism operating at the equator outside the plasmasphere, but only on small pitch-angle particles; (2) a mechanism operating off the equator, acting on all the particles it can reach (thus excluding the large pitch-angle particles which will already have mirrored). The author knows of no plausible mechanism of type (1), although he would be glad to hear of one. An off-equatorial mechanism is discussed in Section 3.

Next, we turn to data [Williams et al., 1973] supporting the idea of strong wave-particle interactions at the plasmapause. The data comes from $S^3 - A$, launched into an elliptical equatorial orbit with apogee of L = 5.2. It is thoroughly instrumented to explore ring current dynamics, and much useful data are available. Unfortunately, a detector saturation problem makes it difficult to interpret some of the pitch-angle results.

To summarize the data: protons of energy $\lesssim$ 150 keV show a sudden decrease in intensity in the vicinity of the plasmapause (Fig. 7), with the highest energies dropping out first. Protons of energy > 150 keV do not seem to be much influenced by the plasmapause. These facts are consistent with the conventional picture of the EMC instability, according to which the protons must have both sufficient energy and sufficient flux to go unstable.

The energy threshold is given by

$$E \geq E_R = \frac{B^2}{8\pi N} A^{-2} (1 + A)^{-1}$$

where B is the earth's magnetic field, N the total plasma density, and A the pitch-angle anisotropy. The minimum unstable flux is estimated to be $\sim 1 - 5 \times 10^{10}\ L^{-4}\ cm^{-2}\ sec^{-1}$ (lines drawn in on Fig. 7); thus the high-energy protons, with too little flux, might be unaffected. Furthermore, Williams et al. have reconstructed the plasma density N from the resonant-energy formula and their data, identifying E_R at any given L with the energy at which the sudden drop-out occurs in Fig. 7, and using the measured anisotropy. The result of this reconstruction is shown in Fig. 8; both the absolute density values and the shape of the density profile are very reasonable and consistent with the hypothesis of EMC instability.

But three crucial features, which would have further verified this conclusion, are missing: (1) there is no evidence for the requisite ULF wave energy; (2) it is difficult, given the detector saturation problem, to follow the pitch-angle distributions into the plasmasphere, to see if there is a marked decrease in anisotropy; (3) there is, as we have seen, no evidence for strong proton precipitation just inside the plasmapause. In all three cases, it could be that the experiments were not sufficiently sensitive to measure the sought-for effect, but it could also be that an entirely different mechanism is operating. For example, the sudden drop-out of the ring current near the plasmapause might simply reflect the weakening of inward convective transport of protons in the vicinity of the corotation boundary appropriate to the various energies, these boundaries being closer to the plasmapause for lower energies. The lack of precipitation would be simply explained by convective-drift transport to the day side, with ultimate ejection from the magnetosphere. Of course, this leaves the appearance of SAR arcs at the plasmapause unexplained, while the EMC instability theory does yield a reasonable explanation [Cornwall et al., 1971a].

We close with one bit of evidence which is in favor of pitch-angle diffusion of protons in the energy range 100-200 keV [Berg and Søraas, 1972]. These authors observe that protons whose mirror points dip into the atmosphere at a certain longitude are replenished westward of this longitude (the windshield-wiper effect) at a rate consistent with a pitch-angle diffusion coefficient of the order of 10^{-6} sec^{-1}, as might be produced by ULF wave fields of 100-200 milligamma. Such fields produce but weak precipitation, and would be only marginally detectable by many satellite-borne magnetometers.

3. OFF-EQUATORIAL DYNAMICS

What instabilities are there that (1) are strongest off the equator and outside the plasmapause; (2) are capable of profoundly influencing both low-energy protons and protons of energies greater than 100 keV? The EMC mode is not a good candidate; the growth rate is largest inside the plasmasphere and when the Alfven speed is smallest, i.e., at the equator. Anomalous resistance produced by current-driven instabilities in or near the ionosphere may allow potential drops capable of precipitating protons of a few keV [Gregory et al., 1973], but these will not affect 100-keV protons very much.

Let us consider the Post-Rosenbluth [Rosenbluth and Post, 1965] electrostatic mode. This mode is convectively unstable in the presence of a sufficiently large anisotropy of energetic ions, just as for the ion-EMC mode, and has been studied for magnetospheric conditions by Coroniti et al. [1972] and Cornwall and Schulz [1973]. Both these groups conclude that addition of cold ions makes this mode less unstable, so that its primary effect should appear outside the plasmasphere. This is consistent with the previously-mentioned reports of strong precipitation in the same region. We are interested in seeing whether there is any reason to suspect that the PR mode is most strongly unstable off the equator.

This mode exists under the conditions $\Omega_e >> \omega_i >> \Omega_i$; $\omega \simeq \omega_i$; $\omega >> k_{||}\bar{v}_e$; $k_{||}/k_{\perp} \sim (m_e/m_i)^{1/2}$ where Ω_e, Ω_i are cyclotron frequencies, ω_i, ω_e are plasma frequencies based on the total (hot plus cold) plasma density, and $\bar{v}_e$, $\bar{v}_i$ are characteristic velocities. Under these conditions, the electrons are tied to the field lines, while the anisotropic hot ions stream through them on essentially straight-line orbits (on the ion plasma-frequency time scale).

The result is essentially a two-stream instability. The dispersion relation incorporating finite-β effects is

$$1 + b^2 D - \frac{\omega_{ci}^2}{\omega^2} = \frac{\zeta\omega_i^2}{D\omega^2} + \frac{\omega_{Hi}^2}{\omega^2} y^2 F(y) \qquad (1)$$

where we have used the notation

$$b^2 = \omega_e^2/\Omega_e^2 \ , \quad D = 1 + \omega_e^2/c^2k^2 \ ,$$
$$k_{||}/k = (\zeta m_e/m_i)^{1/2} \ , \quad y = \omega/k\bar{v} \qquad (2)$$

and ω_{ci}^2 , ω_{Hi}^2 are the plasma frequencies for the cold and hot ions: $\omega_{ci}^2 + \omega_{Hi}^2 = \omega_i^2$. The function $F(y)$ is given by

$$F(y) = -2 \int_0^\infty dx \, \frac{d\phi}{dx} \left(1 - \frac{x}{y^2} \right)^{-1/2} \qquad (3)$$

$$\phi(v_\perp^2/\bar{v}^2) = \pi \, \bar{v}^2 \int dv_{||} \, f_i(v_\perp^2 , v_{||}^2) \qquad (4)$$

Here f_i is the ion distribution function, normalized to one, and $\bar{v}$ is chosen so that

$$\int_0^\infty dx \, \phi(x) = 1$$

The complex function $F(y)$ has order of magnitude one; for $d\phi/dx > 0$ (positive anisotropy) it is possible for Im F to be greater than zero when Im $\omega > 0$, yielding instability. Maximum growth occurs for Re $y \approx 1$.

In the magnetosphere, b^2 is large at the equator, even outside the plasmasphere, if the density N exceeds 1 cm^{-3}. But b^2 is not necessarily large at moderate-to-high latitudes. The quantity D represents a finite-β correction; simple manipulations of the formulas below show that

$$D \sim 1 + \beta_i (1 + b^{-2})^{-1} \tag{5}$$

where $\beta_i = 4\pi N M_i \bar{v}^2 / B^2$ is the ion β . Thus $D < 1 + \beta_i$ for any b .

Because the PR mode is convective, we take the frequency as real, and calculate a complex $k_{||}$ from the dispersion relation. In so doing, we fix Re ζ to be of O(1). Note that Re $y^2F(y)$ is positive and of order one for Re $y \simeq 1$; then the real part of the dispersion relation tells us that

$$\omega \sim \omega_i \; (1 + b^2 D)^{-\frac{1}{2}} \tag{6}$$

modulo a numerical coefficient of order one; from this and from Re $y \sim 1$, we find

$$k \sim (\omega_i / \bar{v}) \; (1 + b^2 D)^{-\frac{1}{2}} \tag{7}$$

(Using this in the definition of D yields (5).)

Now solve the dispersion relation for Im $k_{||}$, taking into account the fact that the real part of (1) is satisfied:

$$\text{Im} \frac{k_{||}}{k} = \frac{D^{\frac{1}{2}}}{\omega_e} \text{Im} \left[\frac{\omega_i^2}{D} \text{Re}\, \zeta - i\, \omega_{Hi}^2 \; \text{Im}\, y^2 F(y) \right]^{\frac{1}{2}} \tag{8}$$

Let us make the estimates Re $\zeta \sim 1$, Im $y^2F(y) \sim 1/2$ at Re $y \sim 1$, $D \sim 1\text{-}2$. Using these and (7), we can estimate

$$\text{Im}\, k_{||} \sim \left(\frac{\Omega_i}{\bar{v}} \right) (1 + b^{-2})^{-\frac{1}{2}} \frac{\omega_{Hi}^2}{\omega_i^2} \tag{9}$$

modulo a numerical coefficient which may be less than one, but larger than, say, 0.1. This expression reveals the dependence of Im $k_{||}$ on latitude through its dependence on the field strength B , occurring in each of the factors. It is easy to verify that, for a distribution function of the type $(v_\perp^2/B)^N \times$ function

of $(v_\perp^2 + v_{||}^2)$, the mean velocity $\bar{v}$ does not depend on B . Of course, Ω_i is linear in B , and tends to maximize Im $k_{||}$ off the equator. The factor $(1 + b^{-2})^{-\frac{1}{2}}$ decreases off the equator, since b^2 is decreasing, but if b^2 is of order 1 or larger, the change is relatively unimportant. As for the remaining factor, it is often the case that, during storm times outside the plasmasphere, essentially all the plasma is hot, so $\omega_{Hi}^2/\omega_i^2 \sim 1$. Under these circumstances, Im $k_{||}$ will indeed maximize off the equator.

Given that b^2 is large, and given the pitch-angle distribution of the hot ions, it is possible to figure out where maximum growth occurs. Let $\omega_{Hi}^2 \sim B^{-N}$, that is, the hot ions have a $(\sin\alpha)^{2N}$ pitch-angle distribution, and let $\omega_{ci}^2 \sim B^{+M}$ (M = 1 might be expected in quiet times). From (9), Im $k_{||}$ depends on B as

$$\mathrm{Im}\, k_{||} \sim \frac{x^{N+1}}{Rx^{-N} + x^{M}} \tag{10}$$

where $x = B/B_{eq}$ and B_{eq} is the equatorial field strength, and $R > 1$ is the ratio of hot to cold ion density at the equator. A simple calculation shows that (10) is maximum at

$$x = x_c = \left(\frac{R}{N + M - 1} \right)^{1/(N+M)} \tag{11}$$

corresponding to an equatorial pitch angle of $\sin^{-1} x_c^{-\frac{1}{2}}$. We find

$$\frac{(\mathrm{Im}\, k_{||})_{max}}{(\mathrm{Im}\, k_{||})_{eq}} = \left(\frac{R + 1}{R} \right) \left(\frac{N + M - 1}{N + M} \right) x_c \tag{12}$$

The off-equatorial maximum is most pronounced when R is large, and when N + M is close to 1 (both of which effects may be most prominent at storm time). Since the waves do not need long distances to exponentiate several

times ($\mathrm{Im}\, k_{||}\, \ell \sim 10$ for $\bar{v} \sim 2 \times 10^8$ cm sec^{-1}, $\Omega_i \sim 10$ rad sec^{-1} gives $\ell \lesssim$ one Earth radius, from (9)), this off-equatorial maximum, if it even exists, might lead to a fairly well-localized (in latitude) disturbance of the proton ring current. In this case, particles mirroring above this latitude would be nearly unaffected, while all particles mirroring below it would be strongly scattered. Thus the situation envisaged in Fig. (3) could be realized.

ACKNOWLEDGMENTS

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FIGURE CAPTIONS

Figure 1. Precipitating proton profiles during quiet-times, based on OV1-17 data [Mizera, 1973]. The electron profile shows the low-altitude trapping boundary.

Figure 2. Comparison of precipitation flux (solid lines) and equatorial trapped flux (dotted lines) for storm-time [Williams et al., 1973] and quiet-time [Pizzella and Frank, 1971] conditions. Taken from Mizera [1973].

Figure 3. Schematic indication of the equatorial pitch-angle distribution of protons outside the plasmasphere.

Figure 4. Precipitated energy vs. L as found by OV1-17, compared to the energy threshold for EMC instability.

Figure 5. L-values of H arcs and SAR arcs, March 23-24, 1969 [taken from Kleckner and Hoch, 1973].

Figure 6. Profiles of omnidirectional proton flux at the equator. Also shown are two possible values for the minimum unstable flux [taken from Williams et al., 1973].

Figure 7. Reconstruction of the cold plasma density N from the resonant energy equation for EMC instabilities [taken from Williams et al., 1973].

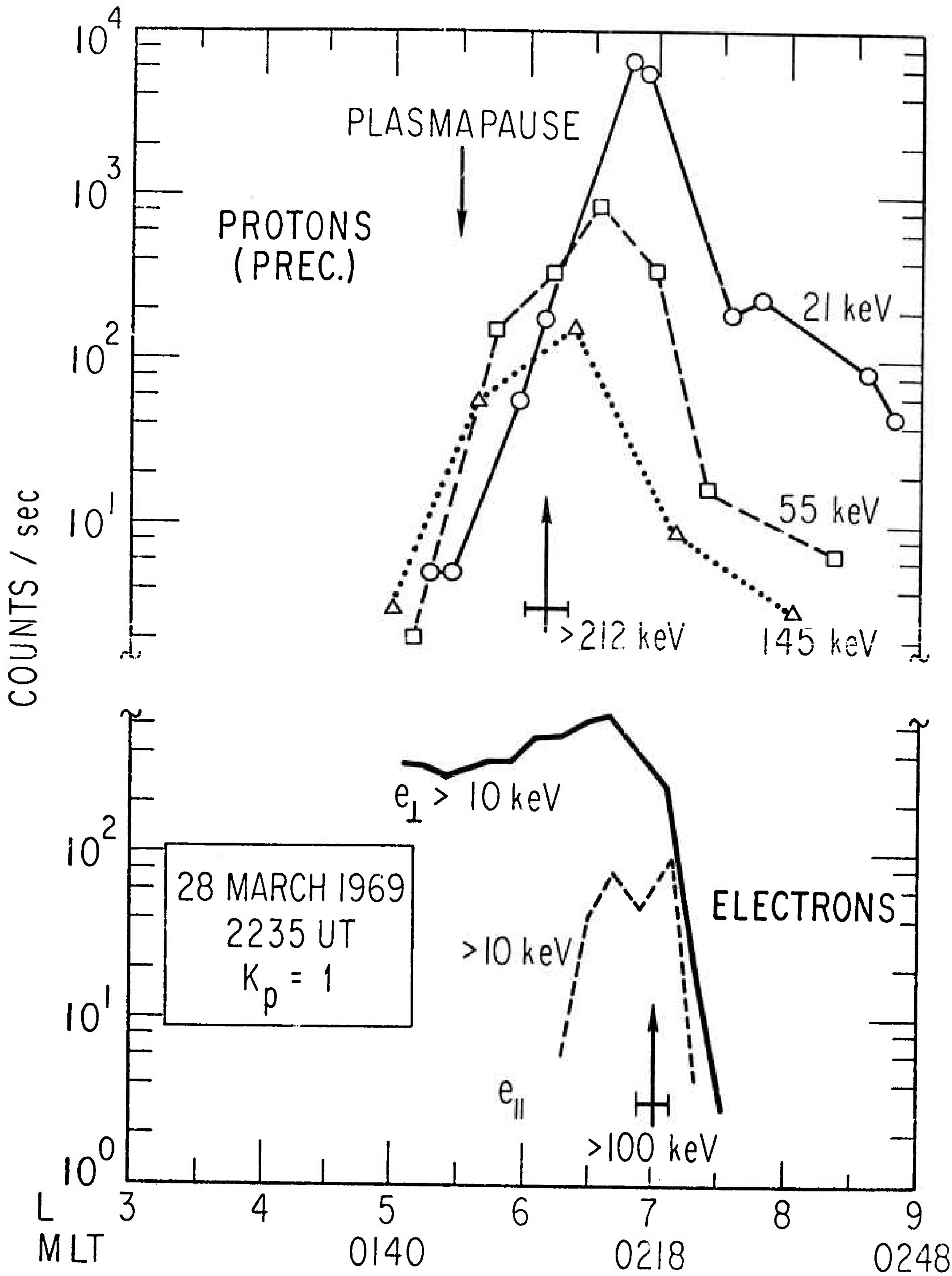
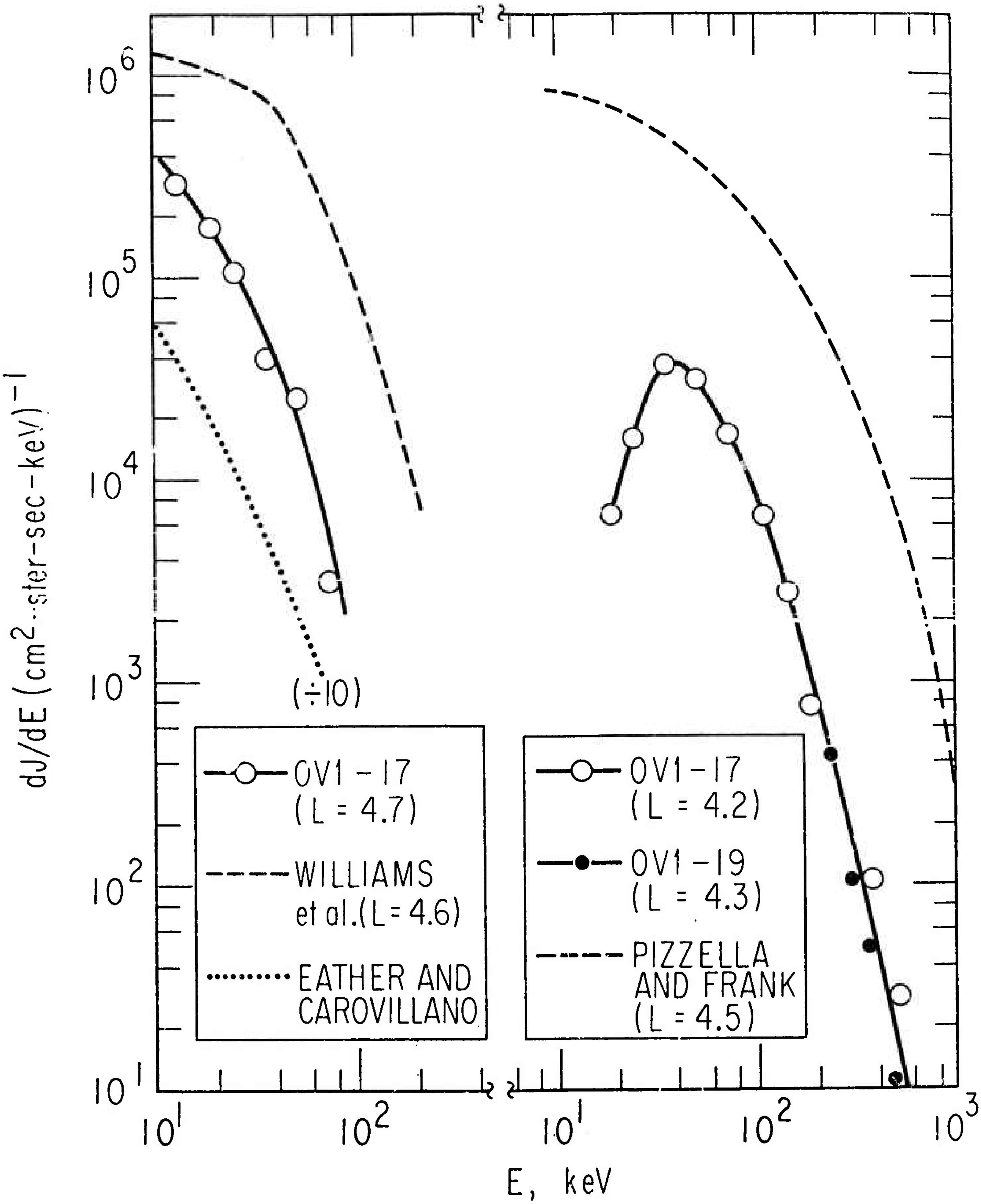

PLASMAPAUSE
PROTONS
(PREC.)
21 keV
55 keV
>212 keV
145 keV
COUNTS / sec
e⊥ > 10 keV
28 MARCH 1969
2235 UT
Kp = 1
ELECTRONS
>10 keV
e∥
>100 keV
L
MLT
0140
0218
0248

Figure 1



10^6
10^5
10^4
10^3
10^2
10^1
dJ/dE (cm^2-ster-sec-keV)$^{-1}$
(÷10)
OV1 - 17
(L = 4.7)
WILLIAMS
et al.(L = 4.6)
EATHER AND
CAROVILLANO
OV1 - 17
(L = 4.2)
OV1 - 19
(L = 4.3)
PIZZELLA
AND FRANK
(L = 4.5)
10^1
10^2
10^1
10^2
10^3
E, keV

Fig. 2

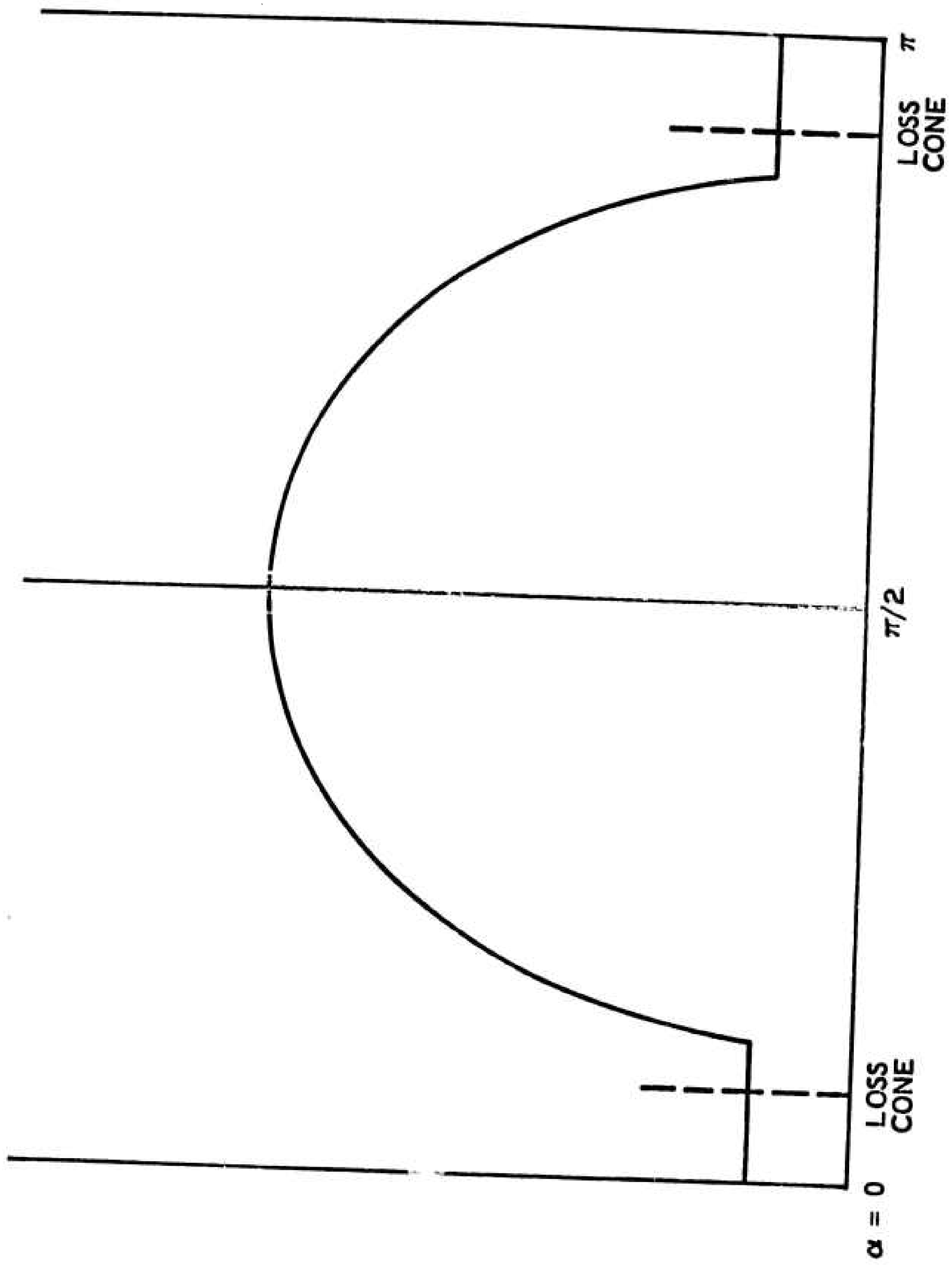
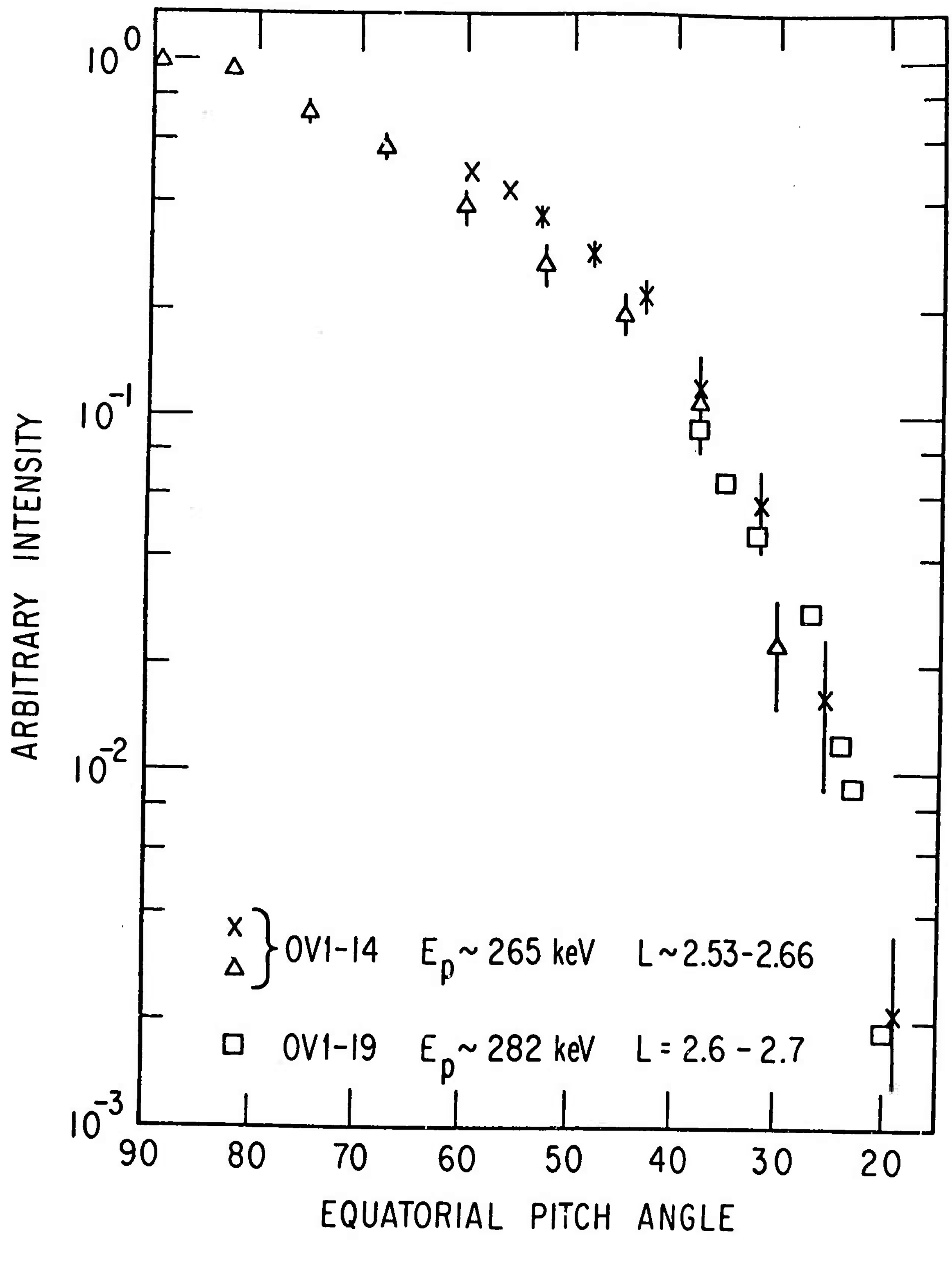
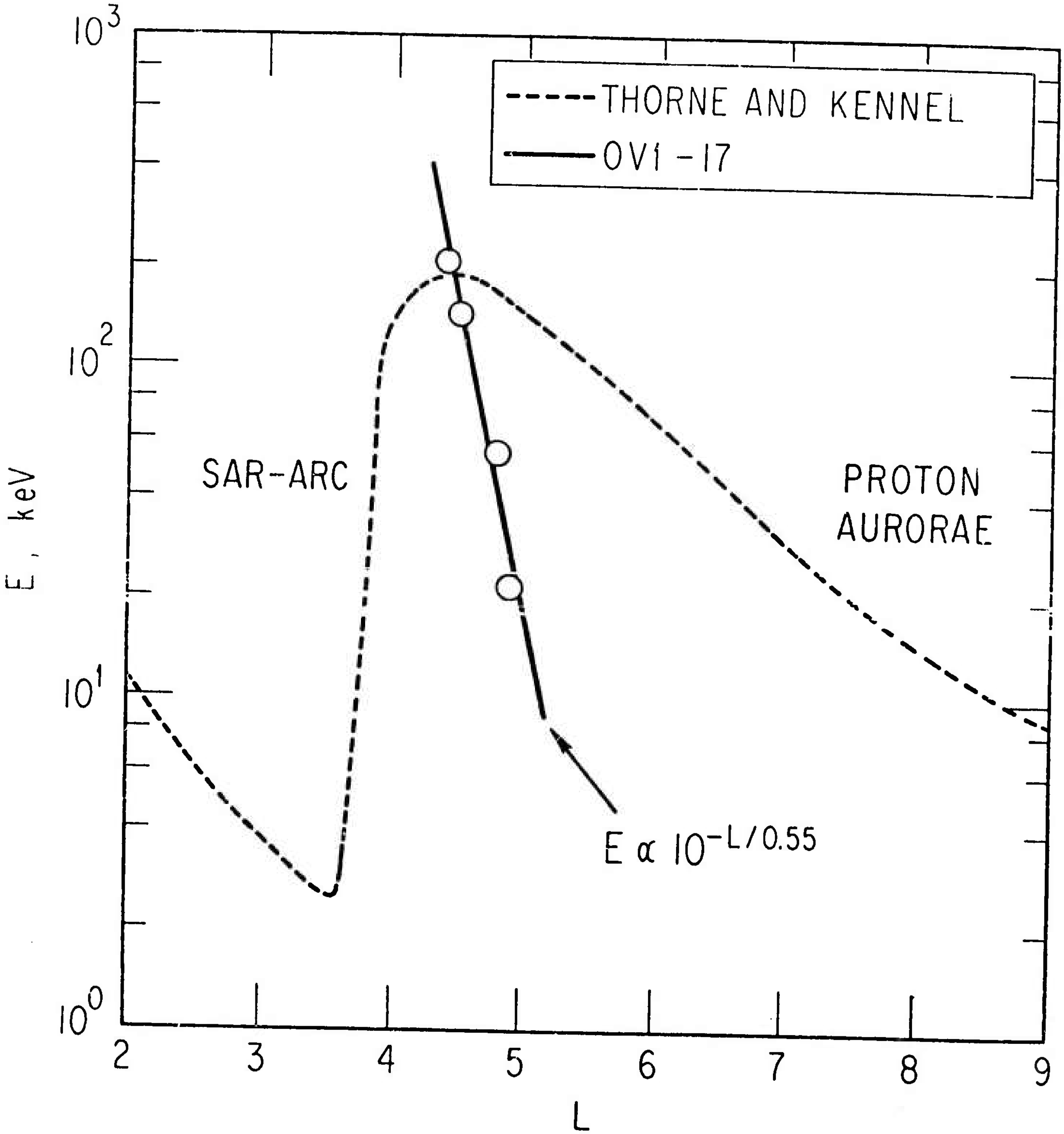


Figure 3



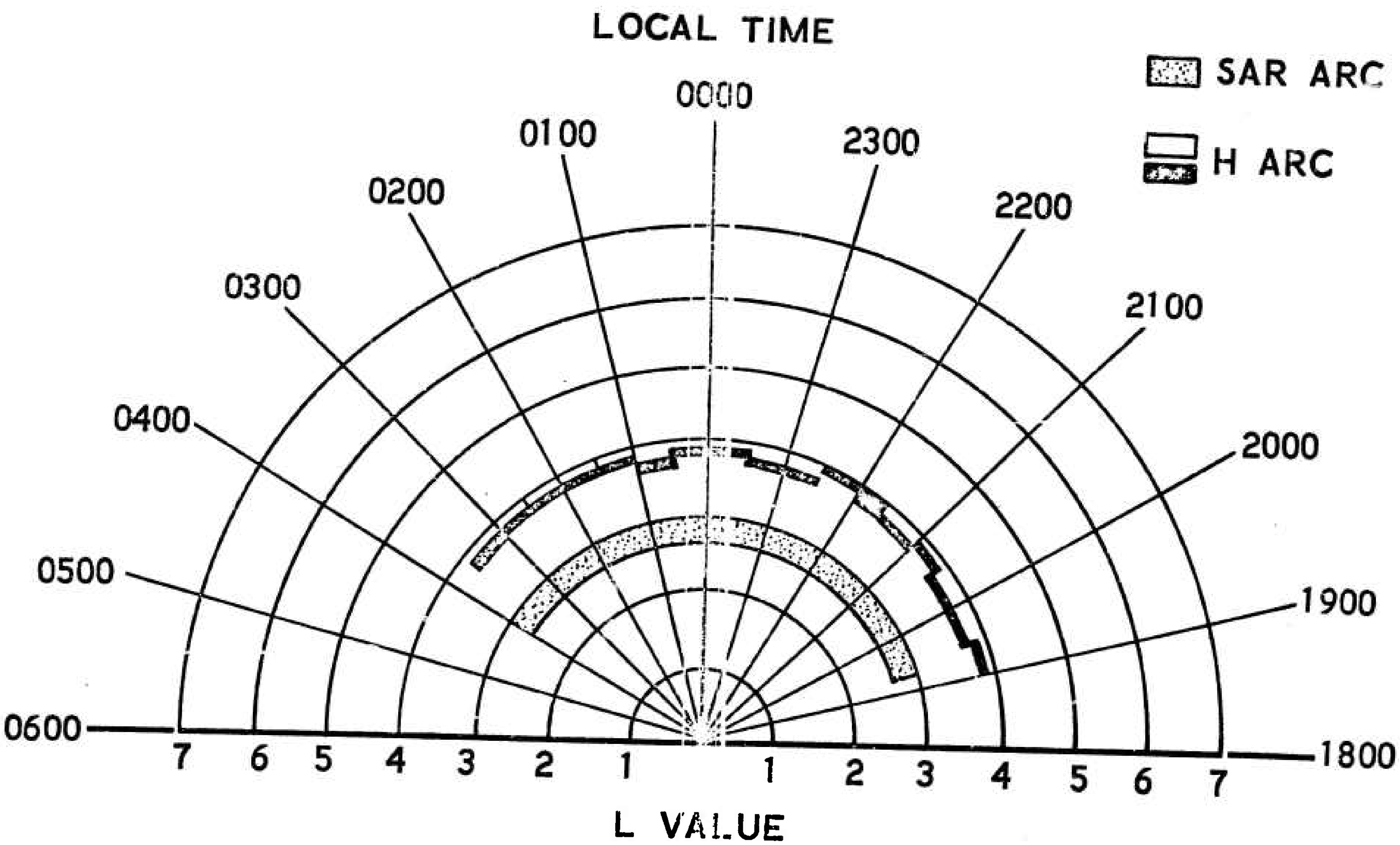
ARBITRARY INTENSITY
10^0
10^{-1}
10^{-2}
10^{-3}
90
80
70
60
50
40
30
20
EQUATORIAL PITCH ANGLE
OV1-14 $E_p \sim 265$ keV $L \sim 2.53-2.66$
OV1-19 $E_p \sim 282$ keV $L = 2.6 - 2.7$

Figure 4



THORNE AND KENNEL
OV1 -17
SAR-ARC
PROTON AURORAE
E α 10^{-L/0.55}
E, keV
L
10^3
10^2
10^1
10^0
2 3 4 5 6 7 8 9

Figure 5



LOCAL TIME
SAR ARC
H ARC
0000
0100
0200
0300
0400
0500
0600
2300
2200
2100
2000
1900
1800
7 6 5 4 3 2 1 1 2 3 4 5 6 7
L VALUE

Figure 6

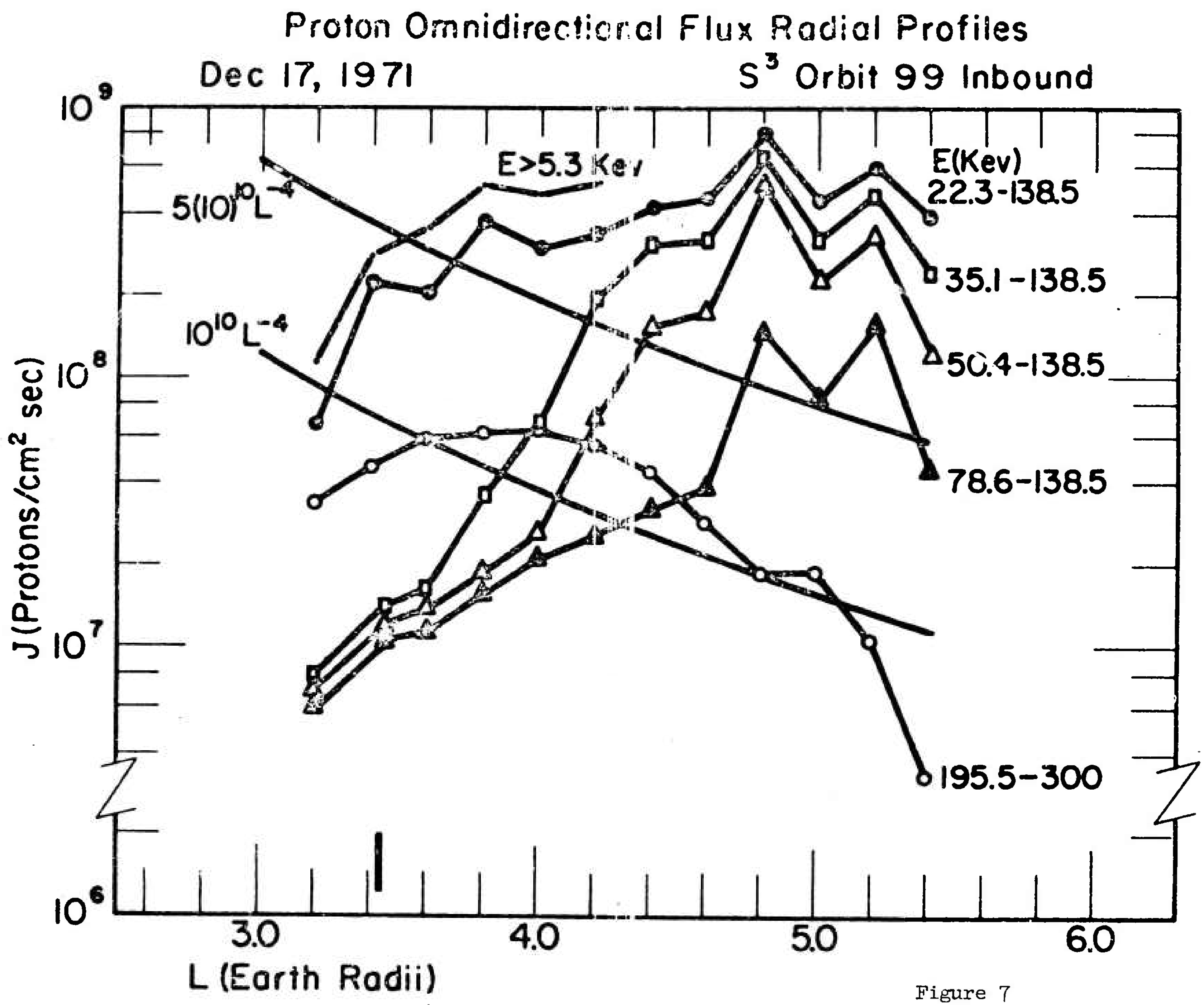

Proton Omnidirectional Flux Radial Profiles
Dec 17, 1971
S^3 Orbit 99 Inbound
10^9
10^8
10^7
10^6
J (Protons/cm^2 sec)
$5(10)^{10}L^{-4}$
$10^{10}L^{-4}$
E>5.3 Kev
E(Kev)
22.3-138.5
35.1-138.5
50.4-138.5
78.6-138.5
195.5-300
3.0
4.0
5.0
6.0
L (Earth Radii)

Figure 7

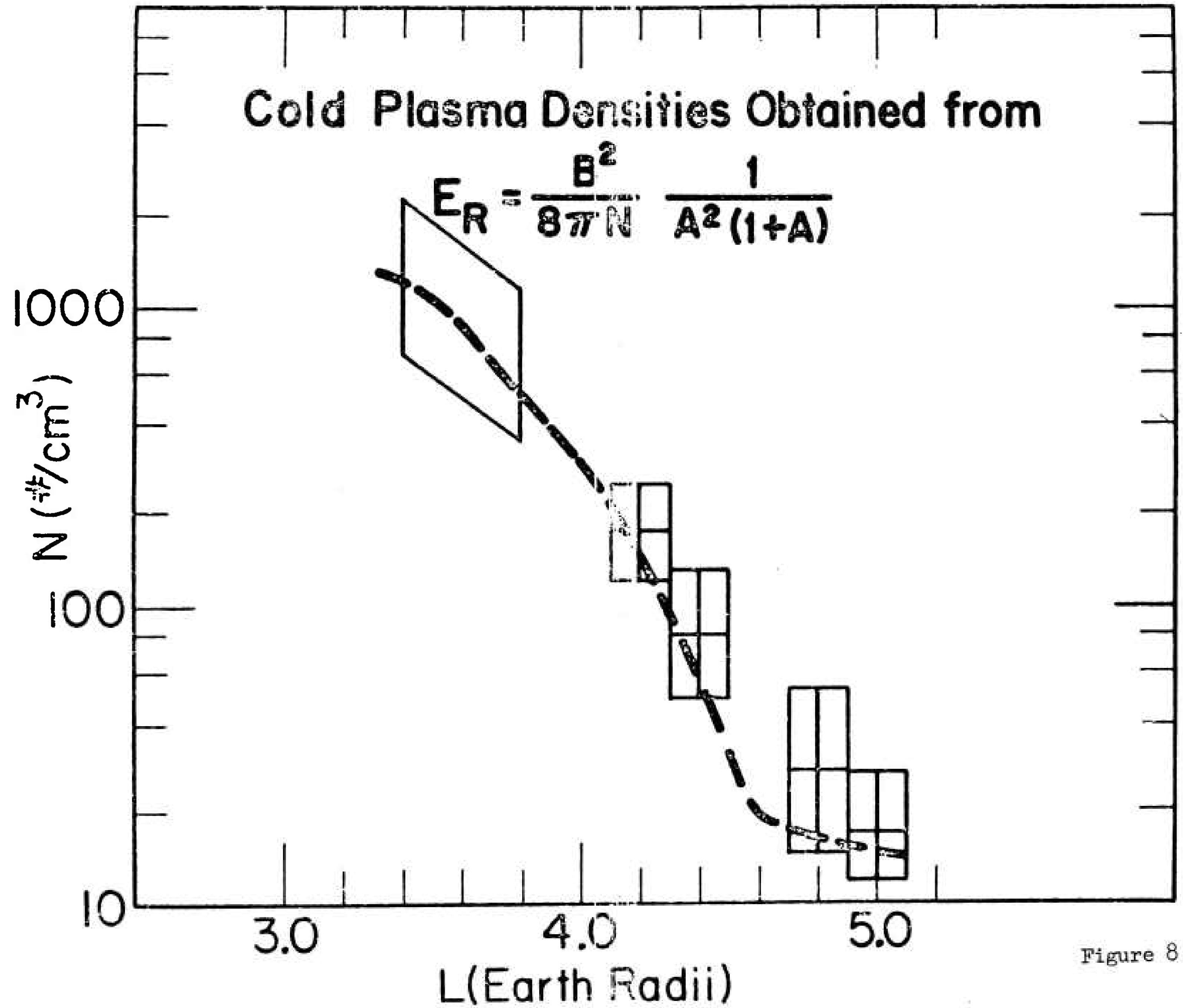

Cold Plasma Densities Obtained from
$E_R = \frac{B^2}{8\pi N} \frac{1}{A^2(1+A)}$
1000
100
10
N (#/cm^3)
3.0
4.0
5.0
L(Earth Radii)

Figure 8

EVALUATION OF LIMIT ON STABLY TRAPPED PARTICLE FLUX

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ABSTRACT

Asymptotic expansions are developed for the self-consistent spectra of protons and electrons (with fixed anisotropy) at marginal stability with respect to their respective electromagnetic cyclotron wave modes. The leading term is the same in each expansion, and implies that $E J_{4\pi}^{*}(E) \sim 10^{10} L^{-4} (B_0/B)^{s}$ cm^{-2}sec^{-1}, where E is the particle energy, $J_{4\pi}^{*}$ is the limiting differential omnidirectional flux, s (~ 1) is the anisotropy, and B/B_0 is the ratio of local B to equatorial B at fixed L. Higher terms in the asymptotic series suggest a very slow convergence, and are essentially useless for quantitative purposes.

INTRODUCTION

It is by now a well-known result of linear plasma theory that anisotropy in a pitch-angle distribution produces instability in the corresponding electromagnetic cyclotron wave mode. Kennel and Petschek (1966) have utilized this principle to place a rough upper bound on the intensity of geomagnetically trapped radiation. There has ensued a great deal of discussion on the significance of such a particle-flux limit.

In its original form, the limit was expressed as an upper bound $I_{4\pi}^*(E^*)$ on the integral omnidirectional flux of electrons having energy in excess of a critical energy $E^* \equiv B^2/8\pi N(s+1)^2 s$, where B is the magnetic-field intensity, N the plasma density, and s the pitch-angle anisotropy. This bound was estimated by Kennel and Petschek (1966) as 7×10^{10} L^{-4} $cm^{-2}sec^{-1}$, assuming evaluation at the geomagnetic equator.

One can readily scale the Kennel-Petschek limit for use at off-equatorial locations by assuming a pitch-angle distribution proportional to $(p_\perp/p)^{2s}$, where p is the particle momentum. It would then follow from Liouville's theorem that

$$I_{4\pi}^*(E^*) \sim 7 \times 10^{10}\, L^{-4}\, (B_0/B)^s\ cm^{-2}sec^{-1}, \tag{1}$$

where B_0 is the equatorial value of B on the field line (L) in question. In this case the particle intensity at off-equatorial points is limited by waves generated in the equatorial region, through which all such particles necessarily pass in the course of their adiabatic bounce motion.

The above considerations have been criticized by Haerendel (1972) and others, who argue that the anisotropy s should not be specified a priori. Since particles that mirror too near the equator, i.e., those having $1 - (B_0/B_m) < E^*/E$, cannot interact with growing waves, there is no limit on the intensity of such particles. By this line of argument, the anisotropy of the pitch-angle distribution should grow without limit, and the critical energy E^* can no longer be defined.

It is essential to recognize, in comparing these seemingly discrepant findings, that Haerendel (1972) and Kennel-Petschek (1966) are not asking the same question. In specifying the anisotropy s a priori, and in calculating a limit on the entire omnidirectional flux, Kennel and Petschek (1966) have tacitly assumed that some separate weak-diffusion process acts to maintain the whole pitch-angle distribution in a form approximating the lowest eigenmode. Haerendel (1972), on the other hand, assumes that there is no such complementary process. The failure to specify underlying assumptions has been a recurring source of needless controversy in the field of radiation-belt physics.

The tacit assumption of Kennel and Petschek (1966) is not at all unreasonable. Lyons et al. (1972), for example, have shown that a parasitic Landau resonance ($\omega = k_\parallel v_\parallel$) with obliquely propagating cyclotron waves can effectively scatter particle pitch angles that are untouched by the fundamental cyclotron resonance ($\omega - k_\parallel v_\parallel = \Omega$). Roberts and Schulz (1968) have shown that resonance with compressional waves at the adiabatic bounce frequency can readily account for the pitch-angle diffusion of particles that mirror very close to the geomagnetic equator.

Taking the Kennel-Petschek assumption (fixed s) as a working hypothesis, it becomes relevant to ask whether the limiting particle intensity can be expressed in terms of a differential spectrum $J^*_{4\pi}(E)$ rather than in terms of a single quantity $I^*_{4\pi}(E^*)$. The answer turns out to be affirmative, but a direct evaluation of $J^*_{4\pi}(E)$ in closed form proves to be cumbersome and generally disappointing. The essential difficulty is that wave-particle interactions do not limit the differential flux itself, but rather a certain integral over the phase-space distribution $\bar{f}$, to which $J_{4\pi}(E)$ is simply related. In a nonrelativistic calculation (Kennel and Petschek, 1966) the bounded integral is given by

$$\int_0^\infty p_\perp \, \bar{f} \, dp_\perp \leq \frac{(c/v_\parallel)^2 \, (\Omega - \omega)^2 \, |\ln R|}{4\pi^3 q^2 \, La \, [s\Omega - (s+1)\omega]}, \tag{2}$$

with $\bar{f}$ having dimensions of $\hbar^{-3}$. The integral is to be evaluated at fixed $v_\parallel$, corresponding to the resonance condition $\omega - k_\parallel v_\parallel = \Omega$, and the indicated bound holds separately on each wave frequency $\omega/2\pi$.

METHOD OF ANALYSIS

From the singularity in (2) it follows that there will be no limit on $J_{4\pi}(E)$ for $E < E^*$, where E^* is the minimum particle energy resonant with a wave having $\omega/\Omega = s/(s+1)$. The resonance condition $\omega - k_\parallel v_\parallel = \Omega$ can be rearranged to read

$$n^2\omega^2(v_\parallel/c)^2 = (\Omega-\omega)^2, \tag{3}$$

where $n\ (\equiv ck_\parallel/\omega)$ is the refractive index. For whistler-mode waves well above the ion gyrofrequency, one obtains $n^2 = \omega_p^{\ 2}/\omega(\Omega-\omega)$, whereupon

$$\omega_p^{\ 2}\omega(v_\parallel/c)^2 = (\Omega-\omega)^3. \tag{4}$$

Inserting $\omega = s\Omega/(s+1)$, one finds

$$E^* = (m/2)v_\parallel^{\ 2} = B^2/8\pi N(s+1)^2 s, \tag{5}$$

where $\omega_p^{\ 2} \equiv 4\pi Nq^2/m$. A similar calculation for ion-cyclotron waves, for which $n^2 = \omega_p^{\ 2}/\Omega_e(\Omega-\omega)$, in resonance with ions yields

$$(\omega_p^{\ 2}/\Omega_e)\,\omega^2(v_\parallel/c)^2 = (\Omega-\omega)^3 \tag{6}$$

and

$$E^* = B^2/8\pi N(s+1)s^2. \tag{7}$$

It is assumed in the foregoing derivations that $n^2 \gg 1$.

A seemingly reasonable approach to the evaluation of $J_{4\pi}(E)$ consists of expanding $J_\perp(E)$ in powers of E^*/E. This method is implemented by taking

$$\bar{f}(p_\parallel, p_\perp) = p^{-2}(p_\perp/p)^{2s}\sum_\ell C_\ell(2mE^*/p^2)^\ell, \tag{8}$$

where $p^2 = p_{\parallel}^2 + p_{\perp}^2$. The omnidirectional flux in this formulation is given by

$$
\begin{aligned}
J_{4\pi}(E) &\equiv 4\pi p^2 \int_0^1 \bar{f}\, dx \\
&= 4\pi \sum_{\ell} C_{\ell} (2mE^*/p^2)^{\ell} \int_0^1 (1 - x^2)^s\, dx \\
&= 2\pi\, B(1/2, s+1) \sum_{\ell} C_{\ell}\, (2mE^*/p^2)^{\ell} \qquad (9)
\end{aligned}
$$

at constant p, where x is the cosine of the (equatorial) pitch angle and B denotes the beta function.

With $\bar{f}\,(p_{\parallel}, p_{\perp})$ given by (8), the bounded integral in (2) is found to be expressible as

$$
\int_0^{\infty} p_{\perp} \bar{f}\, dp_{\perp} = \frac{1}{2} \sum_{\ell} C_{\ell} (2E^*/mc^2)^{\ell} (c/v_{\parallel})^{2\ell} B(\ell, s+1). \qquad (10)
$$

Although (9) and (10) hold in general, it is of interest to determine that particular set of flux coefficients C_{ℓ} for which the bounded integral is equal to its maximum allowed value at each $v_{\parallel}$. The flux coefficients of this particular set (to be denoted C_{ℓ}^*) are obtained by expanding the right hand side of (2) in powers of $(c/v_{\parallel})$. This is achieved by iteratively substituting for ω, as given by (4) or (6).

For example, the electron flux limit is evaluated by substituting

$$
\begin{aligned}
\omega &= \omega_p^{-2} (\Omega - \omega)^3 (c/v_{\parallel})^2 \\
&= \omega_p^{-2} [\Omega - \omega_p^{-2} (\Omega - \omega)^3 (c/v_{\parallel})^2]^3 (c/v_{\parallel})^2 \\
&= \ldots \qquad (11)
\end{aligned}
$$

Such an iterative approach proves to be more convenient than an exact algebraic solution of the cubic equation, for the purpose of expanding the right-hand side of (2) in powers of $(c/v_{\parallel})^2$. One finally obtains

$$\sum_{\ell} C_{\ell}^{*} (2E^{*}/mc^2)^{\ell} (c/v_{\parallel})^{2\ell} B(\ell, s+1)$$

$$= \frac{(c/v_{\parallel})^2 \Omega \,|\ln R|}{2\pi^3 q^2 s\, La} \sum_{n=0}^{\infty} \left[1 - \frac{\omega}{\Omega}\right]^2 \frac{(s+1)^n \omega^n}{s^n \Omega^n}$$

$$\begin{aligned} &= (c/v_{\parallel})^2 \,(\Omega/2\pi^3 q^2 s\, La)\,|\ln R|\, \{1 \\ &\quad - (s-1)(\Omega^2/s\omega_p^2)(c/v_{\parallel})^2 \\ &\quad + \tfrac{1}{2}(3s^2 - 3s + 1)(\Omega^2/s\omega_p^2)^2 (c/v_{\parallel})^4 \\ &\quad - \tfrac{1}{6}(2s-1)(6s^2 - 3s + 1)(\Omega^2/s\omega_p^2)^3 (c/v_{\parallel})^6 \\ &\quad + \ldots \}, \end{aligned} \tag{12}$$

whereupon

$$\begin{aligned} J_{4\pi}^{*}(E) &= (s+1)(cB/2\pi^2 qEs\, La)\,|\ln R|\, B(1/2, s+1)\, \{1 \\ &\quad - (s-1)(s+2)(s+1)^2 (E^{*}/E) \\ &\quad + \tfrac{1}{2}(3s^2 - 3s + 1)(s+3)(s+2)(s+1)^4 (E^{*}/E)^2 \\ &\quad - \tfrac{1}{6}(2s-1)(6s^2 - 3s + 1)(s+4)(s+3) \\ &\qquad \times (s+2)(s+1)^6 (E^{*}/E)^3 \\ &\quad + \ldots \}. \end{aligned} \tag{13}$$

This asymptotic expansion of the limiting electron spectrum $J^*_{4\pi}(E)$ has the disagreeable property of being very slowly convergent. The expansion must (of course) diverge for $E \leq E^*$, but one might have hoped for rapid convergence where $E \geq 10\,E^*$. Instead, the series in curly brackets reads $1 + (45/16)(E^*/E) + (2835/512)(E^*/E)^2 + \ldots$ for $s = 1/2$ and $1 + 96\,(E^*/E)^2 - 2560\,(E^*/E)^3 + \ldots$ for $s = 1$.

The situation is not greatly different in the case of ions resonant with ion-cyclotron waves, except that the iterative substitution is

$$\omega = \Omega\,[1 - (\omega/\Omega)]^{3/2}\,(c_A/v_{\parallel}), \tag{14}$$

where c_A is the Alfvén speed. It follows that ℓ must take on half-integer values as well as integer values. The limiting spectrum $J^*_{4\pi}(E)$ is found to have the asymptotic expansion

$$\begin{aligned} J^*_{4\pi}(E) = {} & (s+1)(cB/2\pi^2 qEsLa)\,|\ln R|\,B(1/2,\, s+1)\,\{\,1 \\ & - 2(s-1)(s+1)^{1/2}\left[\Gamma(s+\tfrac{5}{2})/\Gamma(s+2)\right](E^*/\pi E)^{1/2} \\ & - \tfrac{1}{2}\,(3s^2 - 3s - 2)(s+2)(s+1)(E^*/E) \\ & + (\pi/6)(21s^3 - 21s^2 - 6s + 8)\left[\Gamma(s+\tfrac{7}{2})/\Gamma(s+2)\right] \\ & \qquad \times (s+1)^{3/2}(E^*/\pi E)^{3/2} \\ & - \tfrac{1}{4}\,(10s^4 - 10s^3 - 17s^2 + 5s - 2)(s+3) \\ & \qquad \times (s+2)(s+1)^2(E^*/E)^2 \\ & + \ldots\} \end{aligned} \tag{15}$$

in this case. The series in curly brackets reads $1 + (8/\pi\sqrt{6})(E^*/E)^{1/2} + (165/32)(E^*/E) + \ldots$ for $s = 1/2$ and $1 + 6(E^*/E) + \ldots$ for $s = 1$.

In summary, neither the electron case nor the proton case yields a rapidly convergent expansion for $J_{4\pi}^*(E)$. One may as well accept the leading term of each series as an indicator of the saturated spectrum. In each case the leading term yields

$$J_{4\pi}^*(E) \sim (s+1)(cB/2\pi^2 q E s La)\,|\ln R|\, B(1/2,\ s+1) \tag{16}$$

for $E \gg E^*$. It follows asymptotically for reasonable values of s ($s = 1/2$ or $s = 1$) that

$$E J_{4\pi}^*(E) \sim 10^{10}\, L^{-4}\, \mathrm{cm}^{-2}\, \mathrm{sec}^{-1}. \tag{17}$$

This result constitutes about the best available nonrelativistic estimate for the saturated particle spectrum. The generalization to off-equatorial points would be that

$$E\, J_{4\pi}^*(E) \sim 10^{10}\, L^{-4}\, (B_0/B)^s\, \mathrm{cm}^{-2}\, \mathrm{sec}^{-1}, \tag{18}$$

where B_0 is the equatorial value of B on the field line (L) in question. The incompatibility between (18) and (1) arises from the specification of a spectrum steeper than E^{-1} in the derivation of $I_{4\pi}^*(E^*)$ by Kennel and Petschek (1966). The self-consistent spectrum turns out proportional to E^{-1} in the asymptotic limit, although relativistic effects not yet evaluated may alter this result.

DISCUSSION

The present attempt to evaluate $J_{4\pi}^{*}(E)$ for fixed s has been largely unsuccessful, in that it has yielded only a pair of slowly convergent asymptotic expansions. One may as well invoke (18) at all energies above E^{*}, since the higher terms in the two asymptotic expansions are practically useless in the quantitative sense. A cut-off at some high energy may be in order, both because the present calculation is nonrelativistic and because magnetospheric particle sources do not extend to arbitrarily large E. While it is true that a cut-off energy $\sim e^{7}E^{*}$ would reconcile (18) with (1), this should not be a major consideration.

ACKNOWLEDGMENTS

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PARTICLE LIFETIMES IN STRONG DIFFUSION

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ABSTRACT

The mean lifetime τ of a particle distribution, driven to isotropy by intense pitch-angle diffusion, is calculated by analytical means for conditions applicable to the earth's magnetosphere. The resulting algebraic expressions reduce to $\tau \approx [64 La/35v\alpha_c^2(1-\eta)]$ in the limit of a small equatorial loss cone (half-angle α_c), where v is the particle speed, L is the magnetic shell parameter, $\underline{a}$ is the radius of the earth, and η is the particle albedo from the atmosphere at either foot of the field line. Distinction is made in the full expressions for τ between complete isotropy (caused by strong pitch-angle diffusion all along the field line) and incomplete isotropy (caused by strong diffusion that is localized at the magnetic equator) over the upward hemisphere in velocity space.

INTRODUCTION

As described by Kennel (1969), strong diffusion is the consequence of having a bounce-averaged pitch-angle diffusion coefficient D_{xx} ($x \equiv$ cosine of equatorial pitch angle) much larger than $\alpha_c^2 \Omega_2$, where $\Omega_2/2\pi$ is a particle's energy-dependent bounce frequency in the geomagnetic field and α_c is the half-angle of the equatorial loss cone. In this limit the mean lifetime of a particle against pitch-angle diffusion into the atmosphere approaches a minimum value τ that is independent of the magnitude of D_{xx}, but sensitive to the magnitude of α_c^2. This is in contrast to the weak-diffusion limit ($D_{xx} \ll \alpha_c^2 \Omega_2$), in which the particle lifetime would be proportional to $1/D_{xx}$.

Kennel (1969) estimated the magnitude of τ as $\pi/\Omega_2 \alpha_c^2$ for $\alpha_c^2 \ll 1$, by reasoning that a particle in either loss cone (each having solid angle $\pi\alpha_c^2$ out of its respective hemisphere, 2π) will be lost within a quarter bounce period after traversing the equator. Lyons (1973) has recently refined the calculation of τ (at least for $\alpha_c^2 \ll 1$) by more carefully specifying the probability that a particle whose guiding center lies within a given magnetic field tube is actually in the loss cone. Lyons evaluated this probability as $1.1 \sin^2\alpha_c$ (rather than Kennel's $\alpha_c^2/2$) by calculating the relative amounts of phase space inside and outside the loss cone seen by the equatorial pitch-angle distribution of particles having the same speed v. By assigning a loss time of π/Ω_2 (rather than Kennel's $\pi/2\Omega_2$) and taking a properly weighted (by the factor x/Ω_2) average of Ω_2 over the equatorial pitch-angle distribution, Lyons (1973) thus obtained $\tau = 2La/1.1\, v \sin^2\alpha_c$, where L is the magnetic shell parameter and <u>a</u> is

the radius of the earth. The weighting factor x/Ω_2 enters because it is proportional (at constant v and L) to the Jacobian of the transformation from canonical phase space to the variables v, x, and L (e.g., Roederer, 1970). Procedures for averaging over the trapped-particle distribution are reasonably straightforward in the limit $D_{xx} \gg \alpha_c^2 \Omega_2$, since the effect of strong diffusion is to make the equatorial pitch-angle distribution isotropic (Kennel, 1969).

The approach adopted in the present investigation differs somewhat in outlook from that used by Lyons (1973). Here the particle content of a magnetic field tube is calculated explicitly, and is divided by the particle current through the feet of the field tube to determine the strong-diffusion lifetime τ. In this case it is the unweighted local pitch-angle distribution which enters, rather than the weighted equatorial pitch-angle distribution. The two approaches are equivalent in principle, and yield the same result in practice for $\alpha_c^2 \ll 1$. However, the present approach leads naturally to certain conceptual refinements, which are quantitatively significant for $\alpha_c^2 \sim 1$ but cannot readily be introduced in the formulation used by Lyons (1973).

The most important advantage of dealing with the local pitch-angle distribution is the ability to identify (and thereby exclude from averages) those points in phase space that are unoccupied by particles. The purely directional averages performed by Lyons (1973) have the effect of including, at any given time, particle coordinates actually located beneath the surface of the earth. While the contamination of such averages by unoccupied particle coordinates is not significant for $\alpha_c^2 \ll 1$, the present

calculation imposes no such limit on α_c, and therefore provides at least a modest generalization of the strong-diffusion result obtained by Lyons (1973).

BASIC EQUATIONS

Evaluation of τ is made tractable by assuming a dipolar magnetic field (which, however, need not be centered within the earth). Along a given field line (L), the equatorial (minimum) field intensity is denoted B_0. The field intensity at the northern foot (where the field line enters the dense atmosphere) is denoted B_n, and the field intensity at the southern foot is denoted B_s. The details of particle interaction with the dense atmosphere are thereby suppressed (a simplifying approximation). However, it is permissible within the framework of the present analysis to allow for a specular albedo (preserving the incident isotropy) of magnitude η.

The differential particle flux per unit solid angle, incident on a surface normal to $\underset{\sim}{B}$ at either foot of the field tube, is equal to $(1/2\pi)$ $J_{2\pi}(E)$, where $J_{2\pi}(E)$ is the differential flux over the entire downward hemisphere (2π steradians) in velocity space. The particle flux across the foot surface is therefore given by

$$J_{\downarrow}(E) = (1-\eta) J_{2\pi}(E) \int_0^1 \cos\alpha \; d(\cos\alpha) = \frac{1}{2}(1-\eta) J_{2\pi}(E), \qquad (1)$$

where α is the local pitch angle. If dA is the equatorial area of the field

tube, then $[(B_0/B_n) + (B_0/B_s)]\, J_\downarrow(E)\, dA$ is the rate of particle loss (per unit energy) from the field tube, since the cross-sectional area of any magnetic field tube is inversely proportional to B.

It remains to calculate the particle content of the same field tube. There is a slight subtlety here, related to the question of how the intense pitch-angle diffusion is distributed along the field line. If conditions of strong diffusion hold all along the field line, so that the particle flux is completely isotropic (even over the upward hemisphere in velocity space), then the particle density is $(2/v)\, J_{2\pi}(E)$ per unit energy, and the field-tube content is given by

$$C(E) = (2\,La/v)\, J_{2\pi}(E) \int_{\theta_n}^{\theta_s} \sin^7\theta \; d\theta \; dA \tag{2}$$

per unit energy, where s is the coordinate of arc length along the field line and θ is the colatitude measured from the northern magnetic pole. The particular colatitudes θ_n and θ_s correspond to the feet of the field line. Evaluation of the integral in (2) yields

$$\begin{aligned} C(E) = {} & (2/35v)\, La\, J_{2\pi}(E) \Big\{ [1-(B_0/B_n)]^{1/2}\, [16 \\ & + 8(B_0/B_n) + 6(B_0/B_n)^2 + 5(B_0/B)^3] \\ & + [1-(B_0/B_s)]^{1/2}\, [16 + 8(B_0/B_s) \\ & + 6(B_0/B_s)^2 + 5(B_0/B_s)^3] \Big\}\, dA, \end{aligned} \tag{3}$$

where a is the radius of the earth.

It proves convenient to introduce the abbreviated notation $B_0/B_n \equiv y_n^2 \equiv 1 - x_n^2$ and $B_0/B_s \equiv y_s^2 \equiv 1 - x_s^2$. The strong-diffusion lifetime τ is thereupon given by

$$\begin{aligned}\tau &= C(E) \div [(y_n^2 + y_s^2)\, J_\downarrow(E)\, dA] \\ &= [4La/35v\,(y_n^2 + y_s^2)(1-\eta)] \\ &\quad \times \{(16 + 8y_n^2 + 6y_n^4 + 5y_n^6)\, x_n \\ &\qquad + (16 + 8y_s^2 + 6y_s^4 + 5y_s^6)\, x_s\} \, . \end{aligned} \tag{4}$$

This expression for τ scales, as it should, like $4La/v$. The bounce period of an individual particle is given by $2\pi/\Omega_2 = (4La/v)\,T(y)$, where $y^2 = B_0/B_m$ and B_m is the mirror-field intensity (attained at $\theta = \theta_m$). However, the function

$$T(y) \equiv \int_{\theta_m}^{\pi/2} [1 - (B/B_m)]^{-1/2} (1 + 3\cos^2\theta)^{1/2} \sin\theta \, d\theta , \tag{5}$$

which contains the explicit dependence of Ω_2 on equatorial pitch angle, does not appear in (4) at all. This is as it should be, since τ is the mean lifetime of a whole distribution of pitch angles. More significantly, and in contrast to prior derivations of τ, the bounce period $2\pi/\Omega_2$ has not entered at any intermediate step between (1) and (4). Given the condition of pitch-angle isotropy, which requires $D_{xx} \gg \alpha_c^2 v/4La$, it is unnecessary to invoke either the bounce frequency or the "probability that a particle is in the loss cone," in deriving a valid expression for τ.

Applied to (4), the limit $y_n^2 = y_s^2 \ll 1$ (whence $y_n^2 \approx \alpha_c^2$) yields $\tau \approx [64\,La/35v\,\alpha_c^2(1-\eta)]$. This result agrees (for $\eta = 0$) with the expression $\tau = 2\,La/1.1\,v\,\sin^2\alpha_c$ given by Lyons (1973).

REFINEMENTS

The assumption of complete pitch-angle isotropy, invoked above in obtaining (2)-(4), seems dubious in reality (see, however, Koons *et al.*, 1972). It requires, in the absence of a unit albedo η, that pitch-angle diffusion low on the field line equalize the upward and downward particle fluxes, *i.e.*, immediately replenish the particle trajectories depleted by entry into the dense atmosphere.

It is perhaps more natural to assume that such particle trajectories are replenished only by the intense pitch-angle diffusion that occurs in the equatorial region, idealized as the point where $B = B_0$. In this limit the upward hemisphere in velocity space would remain depleted along trajectories connecting the equator with the atmosphere. The result is a somewhat shorter lifetime τ than given by (4), since the particle density along the field line is somewhat smaller than the value $(2/v)\,J_{2\pi}(E)$ assumed in deriving (2). The particle content of a field tube is given instead by

$$\begin{aligned} C(E) = (1/v)\,La\,J_{2\pi}(E)\Big\{ &(1+\eta)\int_{\theta_n}^{\theta_s} \sin^7\theta\,d\theta \\ &+ (1-\eta)\int_{\theta_n}^{\pi/2} [1-(B/B_n)]^{1/2}\sin^7\theta\,d\theta \\ &+ (1-\eta)\int_{\pi/2}^{\theta_s} [1-(B/B_s)]^{1/2}\sin^7\theta\,d\theta\Big\}\,dA. \qquad (6) \end{aligned}$$

The first integral has already been performed in obtaining (3).

The second integral in (6), to be denoted $Z(y_n)$, is evaluated by observing [cf. (5)] that

$$dZ/dB_n = (B_0/2B_n^2)\,T(y_n), \tag{7a}$$

or

$$dZ/dy = -y\,T(y). \tag{7b}$$

An excellent approximation for $T(y)$, accurate within 1% for all values of y between 0 and 1, is given by the expression

$$T(y) \approx T(0) + \tfrac{1}{2}\left[T(1) - T(0)\right](y + y^{1/2}) \tag{8a}$$

due to Lenchek et al. (1961). The end-point values are expressible in closed form, viz.,

$$T(0) = 1 + (1/2\sqrt{3})\ln(2+\sqrt{3}) \approx 1.380173 \tag{8b}$$

and

$$T(1) = (\pi/6)\sqrt{2} \approx 0.7404805 \tag{8c}$$

Since (6) implies that $Z(1) = 0$, it follows from (7b) that

$$\begin{aligned} 30Z(y) &= 30\int_y^1 y\,T(y) \\ &\approx (4 - 15y^2 + 6y^{5/2} + 5y^3)\,T(0) \\ &\quad + (11 - 6y^{5/2} - 5y^3)\,T(1). \end{aligned} \tag{9}$$

Since direct evaluation of (6) yields $Z(0) = 16/35 \approx 0.45714$, it would be a good check on the accuracy of (9) to evaluate $Z(0) \approx (4/30)\,T(0) + (11/30)\,T(1)$. Although this approximate expression for $Z(0)$ does not look much like $16/35$, it reduces in fact to about 0.45553 and therefore yields an error of much less than 1%.

Thus, the strong-diffusion lifetime τ under conditions of <u>incomplete</u> isotropy ($D_{xx} \gg \alpha_c^2 \Omega_2$ for all x, but with pitch-angle diffusion localized at $B = B_0$) is given by

$$\begin{aligned}\tau = {} & [2\,La/35\,v\,(y_n^2 + y_s^2)\,(1-\eta)] \\ & \times \{(16 + 8\,y_n^2 + 6\,y_n^4 + 5\,y_n^6)\,(1+\eta)\,x_n \\ & \quad + (16 + 8\,y_s^2 + 6\,y_s^4 + 5\,y_s^6)\,(1+\eta)\,x_s \\ & \quad + 35\,(1-\eta)\,[Z(y_n) + Z(y_s)]\}\,, \end{aligned} \tag{10}$$

with $Z(y_n)$ and $Z(y_s)$ obtained from (9). Of course, the strong-diffusion lifetime is minimized with respect to η by the limit of vanishing albedo ($\eta = 0$), and approaches infinity in the limit of perfect reflection ($\eta = 1$) from the top of the atmosphere. Since $Z(0) = 16/35$, the limit $y_n^2 = y_s^2 \ll 1$ yields $\tau \approx [64\,La/35\,v\,\alpha_c^2(1-\eta)]$ when applied to (10). As noted above, the same limiting expression for τ follows from (4).

DISCUSSION

The foregoing results enable the strong-diffusion lifetime τ for a given field line to be computed, directly from simple algebraic expressions,

in terms of the field intensities at the foot points where the field line enters the dense atmosphere. Although a field line in general enters the atmosphere obliquely, the foot of the field tube (across which particle precipitation occurs) is a surface normal to $\underset{\sim}{B}$. The logic of this contention, invoked above in calculating the particle loss rate, is that the opportunity for precipitation is decided by the location of a particle's guiding center. Precipitation is inevitable (up to a factor of $1-\eta$) once the guiding center has passed a point of no return, such that a particle will lose its energy through ionizing collisions within the next gyroperiod. Thus, the flux $J_{\downarrow}(E)$ calculated in (1) is really a flux of guiding centers as well as a flux of particles. This last interpretation could not be made if the foot of the field tube were not defined as being a surface normal to $\underset{\sim}{B}$.

In the limit of small gyroradii there is little difference between the minimum altitude of a particle and the "perigee" of its guiding center. A particle having appreciable magnetic rigidity, however, can experience a much larger atmospheric density (averaged over gyration) than its guiding center experiences. On the other hand, the gyration-averaged atmospheric density required for precipitation increases with particle energy, as the ionization cross section decreases. Thus, it is not immediately obvious whether the guiding-center altitude that locates the foot of a field line (and thereby determines the parameters B_n and B_s) should be treated as a function (either increasing or decreasing) of the particle energy. Any such energy dependence would be quite weak, and its evaluation would exceed the intended scope of the present work.

In summary, the strong-diffusion lifetime τ has been calculated, for an offset-dipole model of the geomagnetic field, under conditions of complete pitch-angle isotropy (strong diffusion at all values of B/B_0) and incomplete isotropy (strong diffusion localized at $B/B_0 = 1$). The calculation allows for an arbitrary specular albedo η from the foot of the field line (top of the atmosphere). Conditions of incomplete isotropy and vanishing albedo are found to yield the minimum possible particle lifetimes under strong diffusion.

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PARTICLE SATURATION OF THE OUTER ZONE: A NONLINEAR MODEL

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ABSTRACT

Properties of the steady state and transient behavior of geomagnetically trapped radiation are analyzed by means of phenomenological equations that concisely summarize the operative dynamical processes. The equations provide for a realistic coupling between electromagnetic wave energy, particle intensity, and pitch-angle anisotropy in the context of the outer zone. Applications include magnetospheric enforcement of a limit on stably trapped particle flux, the smooth transition between weak pitch-angle diffusion and strong diffusion, parasitic particle precipitation by natural and man-made radio signals, natural and artificial injections of trapped radiation, and the consequences of magnetospheric cold-plasma injection.

INTRODUCTION

Plasma instabilities deriving their growth from non-Maxwellian distributions of particle momenta play an important role in magnetospheric physics. Pitch-angle distributions anisotropic with respect to the ambient magnetic field, for example, are unstable to the generation of electromagnetic cyclotron waves (e.g., Cornwall, 1965; Kennel and Petschek, 1966). Field-aligned auroral currents, corresponding to counterstreaming distributions of ions and electrons, can give rise to a variety of electrostatic (Kindel and Kenne., 1971) and hydromagnetic (Forslund, 1970; Hasegawa, 1970a, b) instabilities. Moreover, electrojet currents transverse to a magnetic field are unstable to ion-acoustic wave generation at remarkably low thresholds in the counterstreaming velocity (Farley, 1963).

There is no major objection to evaluating the linear growth rate of a momentum-space instability for uniform plasma geometry, provided that the magnetosphere is homogeneous on a scale much larger than the wavelength. However, the quasilinear theory of a homogeneous plasma instability, such as might be formulated by following historial precedent (e.g., Vedenov et al., 1962; Rowlands et al., 1966) can fail in many important respects to account for the post-linear evolution of a magnetospheric plasma instability.

Instabilities involving resonant particles (Kennel, 1969), for example, ultimately impel the particle distribution function to form a "plateau" in momentum space, according to the quasilinear theory of a uniform plasma (e.g., Rowlands et al., 1966). Such a plateau assures a vanishing growth rate γ for waves resonant with particle momenta lying on the plateau.

The quasilinear wave spectrum is thereupon calculated by requiring a detailed energy balance between the redistributed particles and the wave vectors $\underset{\sim}{k}$ with which they are resonant. In other words, the idealization of an infinite uniform plasma leads to a simplifying conservation law whereby neither waves nor particles nor energy can enter or leave the system through its boundaries.

Recognition of the magnetospheric plasma as an inhomogeneous medium complicates the quasilinear problem beyond belief. It is natural in this field geometry to express the particle distribution as a function of the three adiabatic invariants. However, the inhomogeneity of the plasma and magnetic field cause a particle to lose resonance with its wave in the course of adiabatic charged-particle motion (e.g., Schulz, 1972). Thus, a particle resonates in turn with a continuously varying succession of wave frequencies in the course of its adiabatic motion. Conversely, an individual wave resonates with a continuously varying set of adiabatic invariants as the wave propagates through the medium. The concept of plateau formation and detailed energy balance seems to be rendered almost meaningless by this complication. In any event, there is no longer a conservation law that prevents waves, particles, and energy from crossing the boundaries of the plasma. Moreover, the usual Fourier decomposition $(\omega, \underset{\sim}{k})$ of a waveform fails because of internal refraction and reflection; a partial solution to this last difficulty is provided by tracing rays (e.g., Thorne and Kennel, 1967) in analogy with the methods of geometrical optics. Even if all the problems involved in formulating the quasilinear theory of inhomogeneous plasmas were cleverly solved from first principles,

however, it is hard to imagine the results being expressed in a form concise enough to be useful for practical applications.

The purpose of the present note is to fabricate instead some heuristic equations that simulate the magnetospheric quasilinear problem in a phenomenologically satisfying way. The electromagnetic cyclotron instabilities (e.g., Cornwall, 1965; Kennel and Petschek, 1966) are chosen in order to make the application definite. The proposed equations are simple in form, relate wave growth to the anisotropy and intensity of the resonant-particle distribution, account for the imperfect internal reflection of wave energy, and recognize the gradual transition from weak pitch-angle diffusion to strong diffusion (e.g., Kennel, 1969) as the scattering time varies relative to the particle bounce period. The equations are directly applicable to a variety of magnetospheric phenomena involving the cyclotron-resonance instabilities, and they lead to simple predictions that can readily be compared with the rudimentary observations normally available.

On the other hand, the equations are not necessarily "true," according to the standards usually recognized in the axiomatic formulation of physical theory. They are instead heuristic phenomenological equations, to be viewed as a prototype of the structure that a properly formulated theory should ultimately reveal. By design, such equations gloss over subtleties of mathematical definition, and the terms therein represent somewhat nebulous averages of ideal physical quantities. However, the observational data themselves often represent somewhat nebulous averages of the real physical quantities.

BASIC EQUATIONS

According to Kennel and Petschek (1966), there is a limiting value I^* that the integral omnidirectional flux $\underline{I}$ (of magnetospherically trapped particles above a certain energy threshold) cannot exceed without provoking instability. In a recent review by Schulz and Lanzerotti (1973), equatorial evaluation of the relevant parameters is found to yield $I^* \sim 10^{11} L^{-4}$ $cm^{-2} sec^{-1}$. For fixed anisotropy of the particle distribution in pitch angle, the linear growth rate γ for the amplitude of an electromagnetic cyclotron wave is proportional to I/I^*. The growth rate for the wave energy (squared amplitude) is 2γ. An incipient wave undergoes partial reflection (coefficient R) upon traveling a distance $\sim La$, where L is the magnetic shell parameter and $\underline{a}$ is the radius of the earth. The remaining fraction $1 - R$ of the wave intensity is lost. The time interval between reflections is estimated as La/v_g, where v_g ($\equiv |d\omega/dk|$) is the group velocity. Thus, the condition for marginal stability (Kennel and Petschek, 1966) is

$$R \exp(2\gamma La/v_g) = 1 \tag{1a}$$

or

$$\gamma = (v_g/2La)|\ln R| \equiv \gamma^*. \tag{1b}$$

If $\underline{I}$ ever exceeds I^*, *i.e.*, if γ ever exceeds γ^*, the consequence is a net growth of wave energy and of particle-loss (precipitation) rate. However, since the absorbing atmosphere is distant from the site of the wave-particle interaction, the actual loss rate can never exceed $1/\tau$, where τ is the strong-diffusion lifetime realized for a particle distribution driven to pitch-angle isotropy by the wave-particle interaction (Kennel and Petschek, 1967).

The foregoing considerations suggest an equation of the form

$$\frac{dI}{dt} = -\frac{\lambda I}{1 + \lambda\tau} + S \qquad (2)$$

to describe the evolution of particle intensity I in the presence of a particle source S. The parameter λ is essentially the pitch-angle diffusion coefficient, modified by appropriate numerical factors so as to make a reciprocal lifetime (Roberts, 1969). In weak diffusion the particle intensity decays as $-\lambda I$ per unit time. In the transition to strong diffusion, the decay rate remains between λ and $1/\tau$.

The factor $1/(1 + \lambda\tau)$ is a rough measure of the residual anisotropy in the pitch-angle distribution. In weak diffusion ($\lambda\tau \ll 1$) the particles assume a natural anisotropy of order unity. The anisotropy is substantially reduced for $\lambda\tau \gtrsim 1$, and this leads to a reduction of the growth rate from that applicable in weak diffusion. Following the above reasoning, it is convenient to estimate that

$$\frac{1}{\lambda}\frac{d\lambda}{dt} = \frac{2\gamma^{*}(I/I^{*})}{1 + \lambda\tau} + \left|\frac{d\omega}{dk}\right| \frac{\ln R}{La} + \frac{W}{\lambda} \qquad (3)$$

in the presence of an external wave source having strength W. The wave intensity derived from W leads to "parasitic" pitch-angle diffusion of the particles (e.g., Kennel and Petschek, 1969). The factor $1 + \lambda\tau$ reduces the growth rate γ from the value $(I/I^{*})\gamma^{*}$ that would hold for the natural (weak-diffusion) anisotropy of order unity. The growth rate

approaches γ^* for $I = I^*$ and $\lambda\tau \ll 1$. The leading factor of 2 enters because λ is proportional to the wave intensity (squared amplitude).

Choice of the divisor $1 + \lambda\tau$ is somewhat arbitrary in (3). A divisor of the form $(1 + \lambda\tau)^{\alpha}$ would seem equally reasonable, where α is any number lying roughly between 1/2 and 4. The choice of $\alpha = 1$ in (3) is motivated by algebraic convenience and justified on the grounds that alternative choices for α would lead to very similar "physical" consequences.

The idealized equations (2) and (3) represent a bounce-averaged treatment of the wave-particle interaction, in that both wave growth and wave reflection are treated as continuous (unmodulated) rather than intermittent processes. The equations cannot be expected to generate a bounce-modulated wave intensity under any circumstances. Similarly, equations of this form cannot be expected to yield bounce-modulated particle phenomena such as electron microbursts (e.g., Lampton, 1967).

Of the various algebraic terms appearing in (2) and (3), the term containing $\ln R$ would seem to require further justification. A simple limiting case should suffice. Thus, in the absence of resonant particles ($I = 0$) and waves of external origin ($W = 0$), the wave intensity initially present at $t = 0$ can be sustained only by internal reflection. Since internal reflection is only partial ($0 < R < 1$), the intensity decays with time. It follows from (3) that

$$\lambda(t) = \lambda(0) R^{|d\omega/dk|(t/La)} \tag{4}$$

under these conditions. This is the desired result, since v_g/La represents

the number of reflections per unit time. Each internal reflection preserves a fraction R of the incident wave energy. Equation (3) formulates this effect as if it arose from a continuous process.

STEADY STATE

The foregoing remarks serve to justify (2) and (3) as a credible set of equations to summarize radiation-belt evolution. It remains to illustrate the consequences of (2) and (3) in situations of magnetospheric interest. The steady state ($dI/dt = 0$, $d\lambda/dt = 0$) is perhaps the simplest case from which useful information can be extracted. The algebraic solutions for λ and I in this case are given by

$$\lambda = (S/I^*) + (W/2Y^*) \tag{5a}$$

and

$$I = [1 + (WI^*/2Y^*S)]^{-1} I^* + \tau S, \tag{5b}$$

respectively. In the limit $\tau W/2Y^* \ll \tau S/I^* \ll 1$, it follows that $\lambda \approx S/I^*$ and $I \approx I^*$. This limit corresponds to the situation envisioned by Kennel and Petschek (1966), in which the particle precipitation is neither parasitic nor derived from strong diffusion.

The significance of I^* as a limiting flux holds only in weak diffusion caused by internally generated waves, but the underlying equations admit a far wider range of parameters. A meaningful presentation of the wider range is given in Figure 1, where the dimensionless quantities $\lambda\tau$ (dashed

curves) and I/I^* (solid curves) are plotted as functions of the dimensionless parameter $\tau S/I^*$ for fixed values of $\tau W/2Y^*$. Evidently strong diffusion ($\lambda\tau \gg 1$) always accompanies the condition $I \gg I^*$ in the steady state.

If waves from the external source produce only weak diffusion, i. e., if $\tau W/2Y^* \ll 1$, then it follows from (5a) and (2) that

$$\begin{aligned} 1 + \lambda\tau &\approx 1 + (\tau S/I^*) \\ &\approx 1 + [\lambda\tau/(1 + \lambda\tau)] (I/I^*) \end{aligned} \tag{6}$$

in the steady state. This is a quadratic equation having the "physical" solution

$$1/(1 + \lambda\tau) \approx \mathrm{Min}\,(I^*/I,\ 1) \tag{7}$$

for the factor $1/(1 + \lambda\tau)$ that roughly characterizes the anisotropy of the pitch-angle distribution (see above).

A somewhat subjective confirmation of (7) is contained in the data on precipitating protons compiled by Cornwall et al. (1971). There the qualitative anisotropy of the pitch-angle distribution was found to depend (at each L value) solely on the particle intensity. The transition band between clearly anisotropic and virtually isotropic fluxes covered only a factor ~ 4 in total flux, and paralleled a critical (maximum anisotropic) flux value proportional to L^{-4}, as expected. According to (7), the condition $I \gtrsim 4I^*$ should correspond to an observationally negligible anisotropy $\lesssim 1/4$.

IMPULSIVE SOURCE

It is instructive to consider the case in which the steady state is abruptly perturbed by an injection of particles at time $t = 0$. This case is realized when the source terms are such that $W = 0$ and

$$S(t) = S_0\theta(-t) + I_1\delta(t). \tag{8}$$

The constant conditions $S = S_0$ and $W = 0$ existing prior to $t = 0$ (i. e., ever since $t = -\infty$) lead to steady-state values $\lambda = \lambda_0$ and $I = I^* + \tau S_0$ given by (5) for the wave and particle intensities. The impulsive source described by (8) thereupon yields an "initial-value" problem such that

$$\lambda(0^+) = \lambda_0 \equiv (S_0/I^*) \tag{9a}$$

$$I(0^+) = I^* + \tau S_0 + I_1. \tag{9b}$$

Subsequent evolution of $\lambda(t)$ and $I(t)$ is determined by (2) and (3), with $S(t) \equiv 0$ for $t > 0$.

It is natural to assume $\lambda_0\tau \ll 0.1$, $2Y^*\tau \gtrsim 10$, and $I_1 \gtrsim 4I^*$ in order to make the initial-value problem "physically" significant. In this case the problem separates conceptually into two parts: the growth of the wave intensity and the decay of the particle flux. In the strictly dichotomous picture, the parameter $\lambda\tau$ grows from $\lambda_0\tau$ to ~ 1 on a time scale $t \sim (I^*/2Y^*I_1)|\ln \lambda_0\tau|$ while $\underline{I}$ remains virtually static. Thereupon, in the presence of strong diffusion ($\lambda\tau \gtrsim 1$), the excess particle intensity decays away with a characteristic lifetime $\sim\tau$. The problem is made

complicated (and interesting) by the fact that wave growth and particle decay overlap somewhat in time.

It can be shown from (2) and (3), however, that

$$d \ln \left[\epsilon (I/I^*) - (1 + \lambda\tau)\right] / dt$$
$$= - 2\gamma^* \lambda\tau/(1 + \lambda\tau) \tag{10}$$

for $t > 0$, where $\epsilon \equiv 2\gamma^*\tau/(2\gamma^*\tau - 1) \gtrsim 1$. Thus, the parameter $\lambda\tau$ seeks the time-varying asymptote $\epsilon(I/I^*) - 1$ on a characteristic time scale $\sim (1 + \lambda\tau)/2\gamma^*\lambda\tau$, which is $\lesssim 1/\gamma^*$ for $\lambda\tau \gtrsim 1$. As soon as the asymptotic approximation $\lambda\tau \approx \epsilon(I/I^*) - 1$ becomes valid, it follows from (2) that

$$d \ln \left[\epsilon(I/I^*) - 1\right] / dt \approx - 1/\tau. \tag{11}$$

This means that $\underline{I}$ decays exponentially toward the constant asymptotic value I^*/ϵ, while $\lambda\tau \to 0$ with the same lifetime (τ).

The case of an impulsive source thus leads to an "overshoot" of the particle intensity $\underline{I}$ toward a final value less than I^*. The overshoot is only moderate (since $\gamma^*\tau \gg 1$), but is conceptually important as an essential consequence of the manner in which equations (2) and (3) are coupled. It is interesting in contrast that, according to the quasilinear theory of a uniform plasma (*e.g.*, Rowlands *et al.*, 1966), the particle distribution diffuses directly (without overshoot) toward a momentum-space configuration that corresponds to marginal stability. Moreover, the final state of marginal stability in a uniform (and unbounded) plasma is accompanied by a nonvanishing quasilinear spectrum of wave energy. The present results for a bounded

plasma illustrate some of the rethinking that must be done in order to evaluate the magnetospheric manifestations of a familiar momentum-space instability.

APPLICATIONS

The foregoing results have both practical and conceptual applications. At the practical level, one may reasonably wish to estimate the transient response of the earth's magnetosphere to natural and artificial injections of geomagnetically trapped radiation. The present formalism is most likely to apply in the outer radiation zone, where the medium already approaches saturation ($I \sim I^*$) in its natural state (e.g., Kennel and Petschek, 1966).

If N is the density of cold plasma, the relevant energy threshold E^* for evaluating I and I^* is given by $E^* \sim B^2/8\pi N$. It follows that $E^* \sim$ 40 keV at $L = 6$ if $N \sim 1$ cm^{-3} there. This energy corresponds (for electrons) to a bounce period ~ 1 sec and a strong-diffusion lifetime $\tau \sim 100$ sec at $L = 6$. A weak-diffusion lifetime $1/\lambda_0 \sim 10^5$ sec appears reasonable, from which follows the estimate $\lambda_0 \tau \sim 10^{-3}$. The whistler-mode waves that resonate with 40-keV electrons at $L = 6$ have a transit time $La/v_g \sim 1$ sec between reflections reasonably characterized by taking $\ln R = -3$. In this case it happens that $2\gamma^* \tau \sim 30$, and the condition $I \sim 10 I^*$ leads to wave amplification ~ 10 dB/transit. The use of these numerical parameters in (8)-(11) should reasonably simulate the consequences of energetic particle injection at $L = 6$, where $I^* \sim 8 \times 10^7$ cm^{-2}sec^{-1}.

Direct particle injection at energies $\gtrsim E^*$ is not the only means of effectively enhancing $\underline{I}$. As Brice (1971) has pointed out, the artificial

injection of substantial densities of cold plasma ($E \sim 1$ eV) outside the plasmasphere would have the effect of reducing E^* (by increasing N). For a steeply falling spectrum of energetic particles, evaluation of I at a substantially reduced E^* can easily imply a very large enhancement of the relevant integral flux. For example, if the differential energy spectrum is proportional to E^{-p}, the value of $I(E^*)$ is proportional to N^{p-1}. If $p = 5$, a mere doubling of the cold-plasma density multiplies $I(E^*)$ by a factor of 16. It would appear eminently reasonable to employ the present equations in this context, to estimate the magnetosphere's transient behavior following the sudden injection of an artificial cold plasma.

At the conceptual level, it is clear that the present approach might be taken as the basis for a more general treatment of wave-particle interactions. There can be li[illegible] doubt that the present formulation is only a prototype of the more genera[illegible]atment. Rather than attempt to characterize the entire wave spectrum by a single intensity parameter λ, for example, one might reasonably assign a separate intensity to each eigenfunction of the medium. Rather than summarize the entire particle distribution by an intensity $I(E^*)$ and an anisotropy $1/(1 + \lambda\tau)$, one might reasonably specify the phase-space distribution of particles as a function of the three adiabatic invariants and (perhaps) their conjugate phases (e.g., Schulz and Lanzerotti, 1973). For the present purposes, however, such formal elaboration would largely defeat the analytical simplicity that emerges from a purely heuristic formulation.

A more feasible offshoot of the present work would be the phenomenological description of anomalous resistivity in a bounded plasma. Kindel and Kennel (1971), for example, have identified several electrostatic

instabilities of field-aligned currents in the topside ionosphere. There is only a tenuous dynamical similarity between such counterstreaming instabilities and those driven by anisotropies. However, there is a certain heuristic analogy between the particle flux (I) and the counterstreaming velocity (u) on the one hand, and between the diffusion coefficient (λ) and effective "collision" rate (ν) on the other.

It is quite natural, in pursuit of the analogy, to specify an equation of the form

$$du/dt = (q/m)\, E_{\parallel} - \nu u \tag{12}$$

where ν is proportional to the electrostatic wave intensity, $E_{\parallel}$ is the strength of an imposed electric field, and q/m is the electronic charge-to-mass ratio. The wave intensity would reasonably be determined by an equation of the form

$$d\nu/dt = 2(\gamma^{*}/u^{*})\,(u - u^{*})\nu + W \tag{13}$$

where W is a weak source term. The steady-state solutions of (12) and (13) are given by

$$\nu = (qE_{\parallel}/mu^{*}) + (W/2\gamma^{*}) \tag{14a}$$

and

$$u = [1 + (mu^{*}W/2\gamma^{*}qE_{\parallel})]^{-1}\, u^{*}. \tag{14b}$$

The latter reduces to $u \approx u^*$ in the limit $W/2\gamma^* \ll qE_{\parallel}/mu^*$. The parameters u^* and γ^* can be identified as the instability threshold (u^*) and the damping rate for $u = 0$ (γ^*), respectively. Since the description of anomalous resistance actually lies beyond the intended scope of the present work, further analysis of (12) and (13) is omitted. Moreover, a very incisive treatment of anomalous resistance has recently been given by Coppi and Mazzucato (1971).

HISTORICAL PERSPECTIVE

The present work is not the first to have treated wave-particle nonlinearities heuristically. Practical techniques for solving the mode-coupling equations of weak plasma turbulence have been summarized by Kadomtsev (1965). The phenomenon of resonance broadening by strong plasma turbulence has been analyzed by Dupree (1966) and applied heuristically by Dum and Dupree (1970) to an electrostatic instability in momentum space. In general, the analogies between wave-particle interactions and particle-particle collisions are quite well established.

However, attempts to apply traditional methods of nonlinear plasma physics to the more complicated magnetospheric problem have been quite few in number. An early description of the mutual coupling between electromagnetic wave growth and pitch-angle diffusion was given by Cornwall (1968), who proposed a phenomenological, but analytically transcendental, equation of the form

$$dI/dt = S - (I/\tau) \exp(-I^*/I) \tag{15}$$

to describe the onset of particle saturation in the earth's magnetosphere. This equation, as subsequently applied by Cornwall et al. (1970) to the precipitation of ring-current protons, rather concisely summarizes the results of all previous thought on the nonlinear magnetospheric problem addressed in the present work.

According to (15), the steady-state relationship between intensity and source is given by

$$
\begin{aligned}
\tau S/I^* &= (I/I^*) \exp(-I^*/I) \\
&\approx (I/I^*) - 1 + \ldots
\end{aligned}
\tag{16}
$$

This agrees very well with (5b) for $\tau S/I^* \gg 1 \gg \tau W/2Y^*$. Moreover, the condition for $I \lesssim 0.1\, I^*$, according to (16), is that $\tau S/I^* \lesssim 5 \times 10^{-6}$ (a rather extreme condition, in view of the previous estimate that $\lambda_0 \tau \sim 10^{-3}$ at $L = 6$). Furthermore, expansion of the exponential in (15) for $S = 0$ and $I \gg I^*$ leads to the equation

$$
dI/dt \approx (I^* - I)/\tau \tag{17}
$$

which agrees with (11) for $Y^* \tau \gg 1$. Thus, simple predictions based on (15) do not differ drastically from those based on (2) and (3).

What, then, are the advantages of a formulation summarized by (2) and (3), over the formulation previously existing? In the author's judgment, there are several reasons for preferring the new formulation proposed here. For one, the conceptual foundations of (2) and (3) are more logical than

those underlying (15). After all, it is the growth rate of the wave intensity (and not the instantaneous value of λ itself) that really depends upon the particle distribution function. It follows that (15) could apply in a time-dependent problem only after the wave intensity has evolved to saturation. To describe the transient period, during which the wave intensity grows from a negligible level (λ_0) to its saturated value, one absolutely requires a differentially coupled equation such as (3). At the very least, then, it can be claimed that (2) and (3) offer coverage that is continuous in time, whereas (15) offers coverage that is necessarily interrupted by wave transients.

It is evident also that (15) cannot easily be modified to include parasitic pitch-angle diffusion, which is already included in (2) and (3) by virtue of the extrinsic wave source W. It would be wrong simply to add a second (parasitic) loss term to (15), since nothing would then prevent the total particle loss rate from exceeding the strong-diffusion rate ($1/\tau$) when $I \gg I^*$. Since W appears explicitly in (3), the new model seems especially suited to the description of controlled experiments involving the artificial transmission of wave energy into the magnetosphere for the purpose of causing particle precipitation.

Finally, it should be noted that (2) and (3) are analytically more convenient than (15). The exponential in (15), for example, prevents one from expressing I/I^* as an elementary function of $\tau S/I^*$ in the steady state. Algebraic manipulation is made cumbersome as a result, and almost every interesting case requires extensive numerical analysis. The new model, as given by (2) and (3), may also require numerical analysis for some applications, but the relative simplicity of these basic equations allows considerably more information to be extracted by purely algebraic means.

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FIGURE CAPTION

Figure 1. Normalized steady-state particle and wave intensities (I/I^* and $\lambda\tau$, respectively) as functions of normalized particle-source strength ($\tau S/I^*$) for discrete values of normalized wave-source strength ($\tau W/2Y^*$).

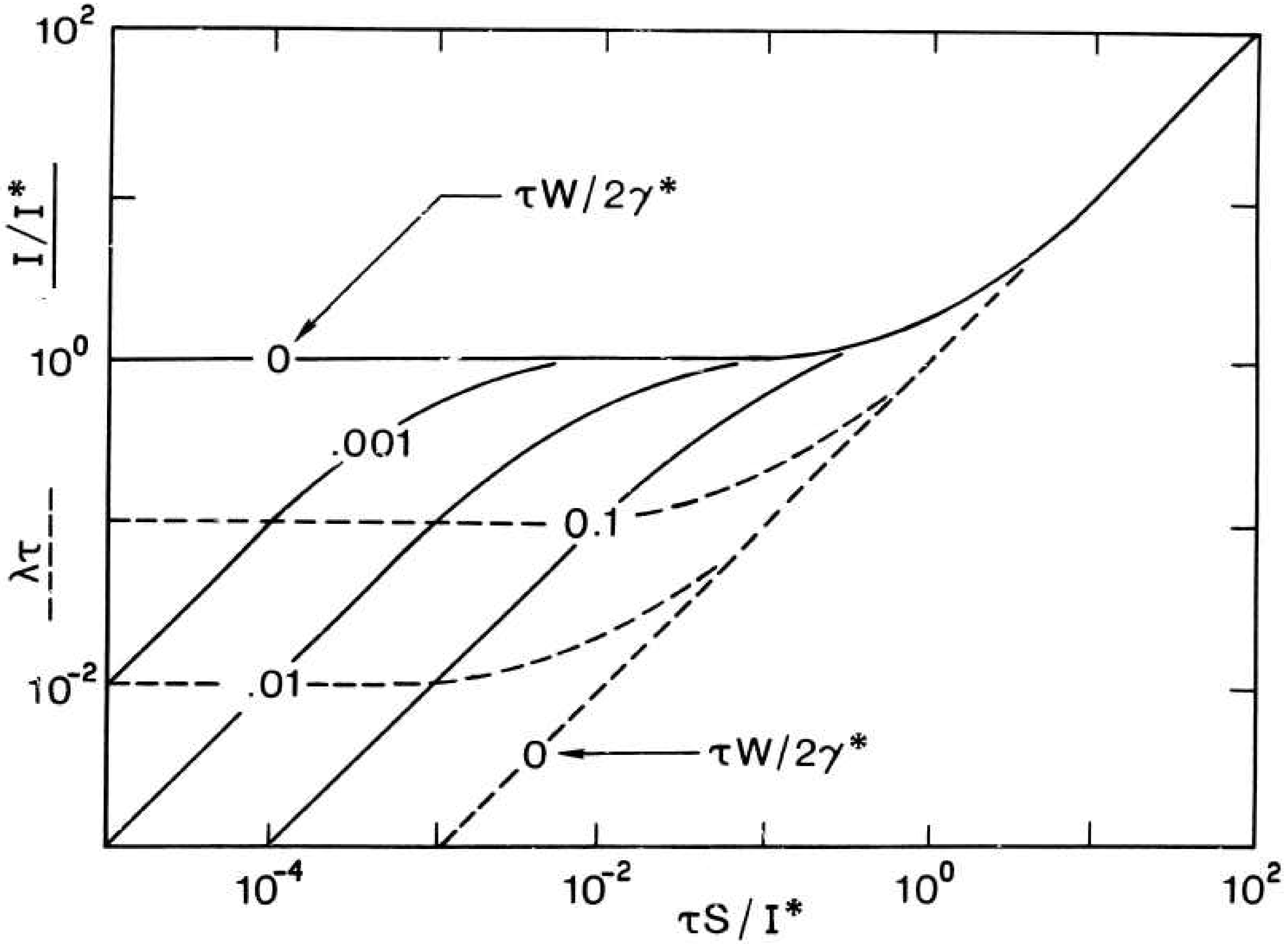
10^2
10^0
10^{-2}
I/I^*
$\lambda\tau$
$\tau W/2\gamma^*$
0
.001
0.1
.01
0
$\tau W/2\gamma^*$
10^{-4}
10^{-2}
10^0
10^2
$\tau S/I^*$

Figure 1

TRAPPING OF PARTICLES BY WAVES IN A NON-UNIFORM PLASMA

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ABSTRACT

The intrinsic bandwidth $\Delta\omega/2\pi$ of equatorial cyclotron resonance, between VLF waves and geomagnetically trapped electrons, is found to exceed the discreteness $\Delta\omega^*/2\pi$ imposed on the underlying wave spectrum by the boundaries of the magnetospheric plasma. Thus, the relevant bandwidth for particle-trapping phenomena (triggered VLF emissions, nonlinear saturation, etc.) is $\Delta\omega/2\pi$ rather than $\Delta\omega^*/2\pi$. The inhomogeneity of the medium permits the attainment of a larger wave intensity at nonlinear saturation than would occur in a uniform plasma having the same equatorial parameters.

INTRODUCTION

The exchange of energy between particles in a plasma and an electromagnetic wave propagating parallel to $\underset{\sim}{B}$ is understood to be inhibited on the scale of the trapping time

$$\tau = (\pi/2)(mc/q b_\perp k_\parallel v_\perp)^{1/2} \equiv \pi/2\tilde{\omega} , \qquad (1)$$

where $b_\perp$ is the amplitude of the wave's magnetic field and $v_\perp$ is the magnitude of $|\underset{\sim}{v} \times \hat{\underset{\sim}{B}}|$ for the typical resonant particle. Thus, a monochromatic wave whose linear growth rate is γ tends to saturate at an amplitude such that $\tilde{\omega} \sim \gamma$.

The saturation of a growing signal at $\tilde{\omega} \sim \gamma$ is clearly at variance with the quasilinear theory of unstable plasma waves (e.g., Rowlands et al., 1966), but very apparent in numerical simulations of plasma-dynamical phenomena (e.g., Ossakow et al., 1972). The failure of quasilinear theory in such computer experiments has been traced by Ossakow et al. (1972) to the discretization of the wave spectrum in $\underset{\sim}{k}$ space. The bothersome quantization of $k_\parallel$ corresponds to the use of periodic boundary conditions in the numerical model. The monochromatic limit thus corresponds to the concentration of wave energy in one or a few values of $k_\parallel$, whereas the quasilinear limit corresponds to the incoherent superposition of wave energy over a broad spectrum of $k_\parallel$ values.

Ossakow et al. (1972) derived from the cyclotron-resonance condition

$$\omega - k_\parallel v_\parallel = \Omega \qquad (2)$$

a velocity separation

$$\Delta v_{\parallel} = (1/k_{\parallel})(v_g - v_{\parallel})\,\Delta k_{\parallel} \tag{3}$$

between particles satisfying (2) for adjacent wave numbers in the discrete spectrum. The velocity bandwidth of cyclotron resonance in a uniform, weakly turbulent plasma i $\gamma/k_{\parallel}$; thus, the criterion for full participation of the velocity distribution in resonant diffusion (as required by quasilinear theory) is

$$\gamma \gtrsim k_{\parallel}\,\Delta v_{\parallel} = (v_g - v_{\parallel})\,\Delta k_{\parallel}\,, \tag{4}$$

where $2\pi/\Delta k_{\parallel}$ is the spatial periodicity imposed on the system. The particle velocity $v_{\parallel}$ and group velocity v_g ($\equiv d\omega/dk_{\parallel}$) are opposite in sign, and so their difference $|v_g - v_{\parallel}|$ is larger than either $|v_g|$ or $|v_{\parallel}|$. Moreover, the condition $\gamma \gg (v_g - v_{\parallel})\Delta k_{\parallel}$ assures that many waves in the spectrum resonate with mutual incoherence on a single particle velocity $v_{\parallel}$. This condition can be important if the resonant particles are to avoid being trapped in the waveform of a discrete signal.

INTRINSIC BANDWIDTH

In a plasma that is non-uniform, cyclotron resonance as defined by (2) is a local and transient phenomenon. In this case, however, the Heisenberg uncertainty principle leads to a nonvanishing bandwidth $\Delta\omega/2\pi$

for the resonant wave frequency. A particle interacting for time t cannot distinguish the carrier frequency of a wave packet to an accuracy better than

$$\Delta\omega/2\pi = t^{-1}\left[1 - (v_{\parallel}/v_g)\right]^{-1}, \tag{5}$$

since the interacting length of wave packet is $t\left[1 - (v_{\parallel}/v_g)\right]$ in time, or $(v_g - v_{\parallel})t$ in space (cf. Roberts, 1968). On the other hand, from the viewpoint of a resonant particle, the ideal value of ω for resonance changes as $\dot{\omega}t$ or $\ddot{\omega}(t^2/8)$ over the interaction time interval (Schulz, 1972).

By equating $\Delta\omega$ with $\dot{\omega}t$ one obtains the (minimum) intrinsic bandwidth for particle cyclotron resonance with a wave spectrum well off the equator. The result is

$$\Delta\omega = \left|2\pi\dot{\omega}\right|^{1/2}\left[1 - (v_{\parallel}/v_g)\right]^{-1/2}. \tag{6}$$

This expression fails to apply near the equator, where $\dot{\omega} = 0$. For equatorial cyclotron resonance one takes $\Delta\omega = \ddot{\omega}(t^2/8)$, whereupon

$$\Delta\omega = \left|(\pi^2\ddot{\omega}/2)\right|^{1/3}\left[1 - (v_{\parallel}/v_g)\right]^{-2/3}. \tag{7}$$

Expressions for $\dot{\omega}$ and $\ddot{\omega}$ are given (Schulz, 1972) by

$$\dot{\omega} = (3c/\omega_p La)\omega^{1/2} (1 + 3 \cos^2\theta)^{-3/2} (3 + 5 \cos^2\theta)$$

$$\times \left[(2 - \nu)\Omega + (\nu + 1)\omega + (\Omega - \omega) \sec^2\alpha\right]$$

$$\times (\Omega + 2\omega)^{-1} (\Omega - \omega)^{3/2} \csc^2\theta \cos\theta \qquad (8)$$

and

$$\ddot{\omega} = \left[(2 - \nu)\Omega + (\nu + 1)\omega + (\Omega - \omega) \sec^2\alpha\right]$$

$$\times (3c/\omega_p La)^2 (\Omega + 2\omega)^{-1} (\Omega - \omega)^3 \qquad (9)$$

for the case of electron-cyclotron resonance with a whistler-mode wave propagating parallel to $\underset{\sim}{B}$. The expression for $\dot{\omega}$ holds at arbitrary colatitude θ, with local pitch angle α. The cold plasma density is taken as proportional to B^{ν}. The expression for $\ddot{\omega}$ applies only at the equator ($\theta = \pi/2$).

It is interesting to consider the $L = 3$ field line, on which $\Omega/2\pi =$ 34 kHz at $\theta = \pi/2$. Estimates for the intrinsic bandwidth $\Delta\omega/2\pi$ are given in Table 1, with cold-plasma densities assumed proportional to B (so that $\nu = 1$). Equatorial densities of 500 cm^{-3}, 1000 cm^{-3}, and 2000 cm^{-3} are regarded as spanning the typical range, and wave frequencies from 3.4 kHz to 30.6 kHz are taken as representative. An equatorial pitch angle of 30° is assumed.

Table 1. Intrinsic Bandwidths (in Hz) of Equatorial Cyclotron Resonance at L = 3

Frequency of Carrier	Density = 500 cm^{-3}	Density = 1000 cm^{-3}	Density = 2000 cm^{-3}
3.4 kHz	22.31	17.71	14.06
6.8 kHz	27.23	21.61	17.16
10.2 kHz	27.57	21.88	17.37
13.6 kHz	25.66	20.37	16.17
17.0 kHz	22.52	17.88	14.19
20.4 kHz	18.65	14.80	11.75
23.8 kHz	14.33	11.37	9.02
27.2 kHz	9.71	7.71	6.12
30.6 kHz	4.92	3.90	3.10

APPLICATION

The existence of an intrinsic bandwidth has been invoked previously in the context of triggered VLF emissions (Schulz, 1972). It is of interest here to explore the effect of an intrinsic bandwidth on the maximum attainable value of $\tilde{\omega}$. Since the effect of $\Delta\omega$ is to impede the trapping of particles by waves, it seems reasonable that $\Delta\omega$ will enable $\tilde{\omega}$ to exceed γ. The amount of excess ought to be a Galilean invariant, which $\Delta\omega$ is not. Thus, it is natural to conjecture that

$$\tilde{\omega} \sim \gamma + [1 - (v_\parallel / v_g)] \, \Delta\omega \tag{10}$$

at monochromatic saturation. This expression defines the maximum amplitude $b_\perp$ to which a sinusoidal signal could grow, viz.,

$$b_\perp \lesssim (mc/qk_\parallel v_\perp) \{ \gamma + [1 - (v_\parallel / v_g)] \, \Delta\omega \}^2 . \tag{11}$$

The corresponding limit in a uniform plasma is obtained from (11) by setting $\Delta\omega = 0$.

The generalization of (11) to a quasi-continuous spectrum $\mathcal{S}(\omega/2\pi)$ is relatively straightforward. One ought to make the identification

$$b_\perp^2/2 = (1/2\pi) \, \mathcal{S}(\omega/2\pi) \, \mathrm{Max}\,(\Delta\omega, \Delta\omega^*), \tag{12}$$

where $\Delta\omega^*$ is the effective discreteness imposed on ω by the boundary conditions. This interpretation is roughly compatible with Ossakow et al. (1972) in the limit $\Delta\omega = 0$.

The value of $\Delta\omega^*$ in the earth's field can be estimated from the line integral of $1/v_g$. The expectation that

$$\Delta\omega^* = 2\pi \div \oint (1/v_g)\, ds \tag{13}$$

follows from quantization of the phase integral (circuit integral of $k_\parallel\, ds$) in steps of 2π, where s is the coordinate of arc length along a field line. For whistler-mode waves in a dipole field one obtains

$$v_g = 2\,(\Omega - \omega)^{3/2}\, \omega^{1/2}\, (c/\omega_p \Omega) \tag{14}$$

and

$$\Delta\omega^* = (\pi c/ L a \Omega_0)\, (\Omega^{\nu}/\omega_p^{\,2})^{1/2}\, \omega^{1/2}$$

$$\div \int_0^{\sin\Lambda} \frac{(1 + 3\cos^2\theta)\Omega^{\nu/2}\, d(\cos\theta)}{(\Omega - \omega)^{3/2}(1 - \cos^2\theta)^3}\,, \tag{15}$$

where Ω_0 is the equatorial gyrofrequency and Λ is the invariant latitude of the field line. The factor $\Omega^{\nu}/\omega_p^{\,2}$ is a constant, since the density is taken as proportional to B^{ν}. Numerical integration of (15) is straightforward, but unnecessary in this instance. It is evident from (15) that

$$\Delta\omega^*/2\pi < (c/2 L a\, \Omega_0)\, (\Omega^{\nu}/\omega_p^{\,2})^{1/2}\, \omega^{1/2}$$

$$\div \int_0^{\sin\Lambda} \frac{(1 + 3\cos^2\theta)\, \Omega^{(\nu-3)/2}\, d(\cos\theta)}{(1 - \cos^2\theta)^3} \tag{16}$$

for $\omega > 0$. Setting $\nu = 1$ and $L = 3$ (whence $\sin^2\Lambda = 2/3$), one obtains

$$\frac{\Delta\omega^*}{2\pi} < \frac{(\Omega c/\omega_p La)_0 (3\omega/\Omega_0)^{1/2}}{3 + \ln(\sqrt{2} + \sqrt{3})} . \tag{17}$$

Since $\omega < \Omega_0$, the right-hand side of (17) cannot exceed 1.12 Hz for a density of 500 cm^{-3}, nor 0.56 Hz for a density of 2000 cm^{-3}. It follows that $\Delta\omega > \Delta\omega^*$ under conditions of interest, at least in this region of the magnetosphere. Resonant particles sample an essentially continuous wave spectrum, since the minimum bandwidth resolvable by a particle exceeds the discreteness of the spectrum. A similar conclusion by Ossakow et al. (1972) was based on the estimate that $\gamma \gg (v_g - v_\parallel)\Delta k_\parallel$. The present result is stronger, in that it does not depend on the magnitude of γ.

DISCUSSION

The significance of $\Delta\omega^*$ would disappear if a wave introduced at one foot of the field line were totally absorbed at the other. At least a partial reflection is required in order to make the wave spectrum discrete by virtue of boundary conditions. In the event of partial reflection of the wave intensity ($0 < R < 1$), there arises the possibility of overall marginal stability (Kennel and Petschek, 1966) if the wave has a local growth rate γ such that

$$|\ln R| = 4La \int_0^{\sin\Lambda} (\gamma/v_g)(1 + 3\cos^2\theta)^{1/2}\, d(\cos\theta). \tag{18}$$

A smaller mean value of γ/v_g would lead to eventual wave attenuation (after perhaps many "hops" along the field line), while a larger mean value of γ/v_g would lead to the spontaneous generation of a finite-amplitude wave signal out of infinitesimal random noise.

The wave spectrum is strictly discrete only at marginal stability. Otherwise the spectrum is characterized by interference "resonances" separated by $\Delta\omega^*/2\pi$ but having definite bandwidths

$$\frac{\Gamma}{2\pi} = \left\{ 4\,La \int_0^{\sin\Lambda} (\gamma/v_g)(1 + 3\cos^2\theta)^{1/2}\, d(\cos\theta) - |\ln R| \right\} \frac{\Delta\omega^*}{4\pi^2} \tag{19}$$

extending to each side of the "resonant" frequencies $\omega^*/2\pi$. Such "resonances" as defined by (13) are not wave-particle interactions, but are closely related to the transmission "resonances" encountered in quantum mechanics (e.g., Merzbacher, 1961). In case $|\Gamma| \gtrsim \Delta\omega^*$, there is considerable overlap of the "resonances", i.e., an essentially continuous spectrum.

As arguments in the previous section have demonstrated, however, considerations on the discreteness of the underlying wave spectrum are substantially overridden in the case of magnetospheric VLF waves by the intrinsic bandwidth of cyclotron resonance between such waves and geomagnetically trapped electrons. Since the intrinsic bandwidth $\Delta\omega$ exceeds the underlying discreteness $\Delta\omega^*$, it matters little whether the underlying

spectrum is essentially discrete ($|\Gamma| \ll \Delta\omega^*$) or essentially continuous ($|\Gamma| \gtrsim \Delta\omega^*$). The spectrum appears essentially continuous to a resonant particle in either case.

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PHASE-INTEGRAL APPROXIMATION OF Pc-4 EIGENFREQUENCIES

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ABSTRACT

The spacing $\Delta\omega/2\pi$ between consecutive toroidal eigenfrequencies $\omega_n/2\pi$ of magnetospheric field lines is correctly given (within 2% at $L = 6.6$) by the reciprocal of $\oint (1/c_A)ds$, where c_A is the local Alfvén speed and s is the coordinate of arc length along the field line. The eigenfrequencies themselves are accurately given by the formula $\omega_n = (n - \delta)\Delta\omega$, where $n = 1, 2, 3, \ldots$ and δ is a number that depends upon the distribution of plasma density (ρ) along the field line. With ρ proportional to r^{-m} (r = radial coordinate), the value of δ at $L = 6.6$ is given by the empirical formula $\delta = 0.44 \{ 1 - \exp[(m - 6)/3] \}$ for $0 \le m \le 6$.

The purpose of this note is to investigate the accuracy of a certain phase-integral approximation for the characteristic frequencies of oscillating magnetospheric field lines. Hydromagnetic (MHD) resonances of this type are believed responsible (e.g., Jacobs, 1970) for the Pc-4 band of geomagnetic micropulsations. A full numerical solution of the eigenvalue problem at $L = 6.6$ (Cummings et al., 1969) makes it possible to check the present formula of approximation against the true eigenfrequencies for a reasonable class of plasma-density models.

In the phase-integral approximation of Pc-4 eigenfrequencies, it is assumed that the oscillation propagates as a noncompressional Alfvén wave ($\omega = c_A k_\parallel$) along the magnetic field line ($r = La \sin^2\theta$) and suffers perfect reflection at the surface of a perfectly conducting earth ($r = a$). Following the spirit of Bohr theory, it is further assumed that the wave action (accumulated over a full bounce period) is quantized in steps of Planck's constant. The classical statement of this principle takes on the form

$$\oint (\omega_n/c_A)\, ds = 2\pi(n - \delta), \tag{1}$$

where n is a positive integer, δ is a constant less than unity, and s is the coordinate that measures arc length along the field line.

In the geomagnetic dipole field, the intensity of $\boldsymbol{B}$ varies with colatitude θ in accordance with the relation

$$B = B_0 (1 + 3 \cos^2\theta)^{1/2} \csc^6\theta \tag{2}$$

at fixed L. The equatorial field B_0 is estimated, for the purpose of comparison with Cummings et al. (1969), as $0.312L^{-3}$ gauss. The plasma density ρ is assumed proportional to $(a/r)^m$, where a is the radius of the earth and m is a constant index. In terms of the equatorial density ρ_0, it follows that

$$\rho = \rho_0 \csc^{2m}\theta \tag{3}$$

at fixed L. The local Alfvén speed c_A is given by $c_A^2 = B^2/4\pi\rho$.

The integral in (1) has a fourfold symmetry. The representation of c_A based on (2) and (3) yields

$$\omega_n/2\pi = (B_0^2/64\pi\rho_0 L^2 a^2)^{1/2}(n-\delta) \div \int_0^{\sin\Lambda} (1-x^2)^{3-(m/2)}\,dx \tag{4}$$

where Λ is the invariant latitude (defined by the identity $L\cos^2\Lambda \equiv 1$). Evaluation of (4) in closed form is straightforward whenever m is an integer (see Appendix).

The eigenfrequencies $\omega_n/2\pi$ computed by Cummings et al. (1969) correspond to a unit equatorial plasma density (i.e., to the case $\rho_0/m_p = 1\ \mathrm{cm}^{-3}$, where m_p is the proton mass) at $L = 6.6$ ($\Lambda \approx 67.1^\circ$). In agreement with (4) the toroidal eigenfrequencies determined by Cummings et al. (1969) are about equally spaced. The mean difference frequency $\Delta\omega/2\pi$ among the first six harmonics is given for each density model (m = 0, 1,

Table 1. Summary of Empirical Data on Toroidal Eigenfrequencies at L = 6.6

Index m	Predicted $\Delta\omega/2\pi$, Hz	Cummings $\Delta\omega/2\pi$, Hz	Cummings $1 - (\omega_1/\Delta\omega)$
0	0.0308	0.0312	0.386
1	0.0287	0.0290	0.355
2	0.0266	0.0268	0.320
3	0.0240	0.0242	0.271
4	0.0213	0.0216	0.214
5	0.0184	0.0186	0.127
6	0.0153	0.0154	0.006

2, . . ., 6) in Table 1. The true values of $\Delta\omega/2\pi$ are thus $\sim$1% larger than the differences

$$\Delta\omega/2\pi = (B_0^2/64\pi\rho_0 L^2 a^2)^{1/2} \div \int_0^{\sin\Lambda} (1 - x^2)^{3-(m/2)}\, dx \tag{5}$$

predicted by (4). Thus, the phase-integral method yields an excellent approximation for $\Delta\omega/2\pi$.

The ratio of the lowest eigenfrequency $\omega_1/2\pi$ to the mean difference $\Delta\omega/2\pi$ provides an empirical determination of $1 - \delta$ within the framework of Cummings _et al._ (1969). The value of δ thus determined from the published period of each lowest eigenmode is given in Table 1 along with the data described above. The empirical values of δ decrease systematically with m. Quantitative interpretation of this trend is complicated by a substantial round-off error ($\sim$0.01) in the evaluation of δ. The resonance frequencies $\omega_n/2\pi$ found by Cummings _et al._ (1969) in the $m = 6$ model, for example, are consistent with the formula $\omega_n/2\pi = 0.01529n$ Hz, with $\delta = 0$.

The empirical values of $\delta = 1 - (\omega_1/\Delta\omega)$ given in Table 1, along with the theoretical values of the trigonometric integral appearing in (4) and (5), are plotted in Figure 1. The empirical formula

$$\delta = 0.44 \{ 1 - \exp[(m - 6)/3] \} \tag{6}$$

is found to hold within the round-off error inherent in Cummings _et al._ (1969).

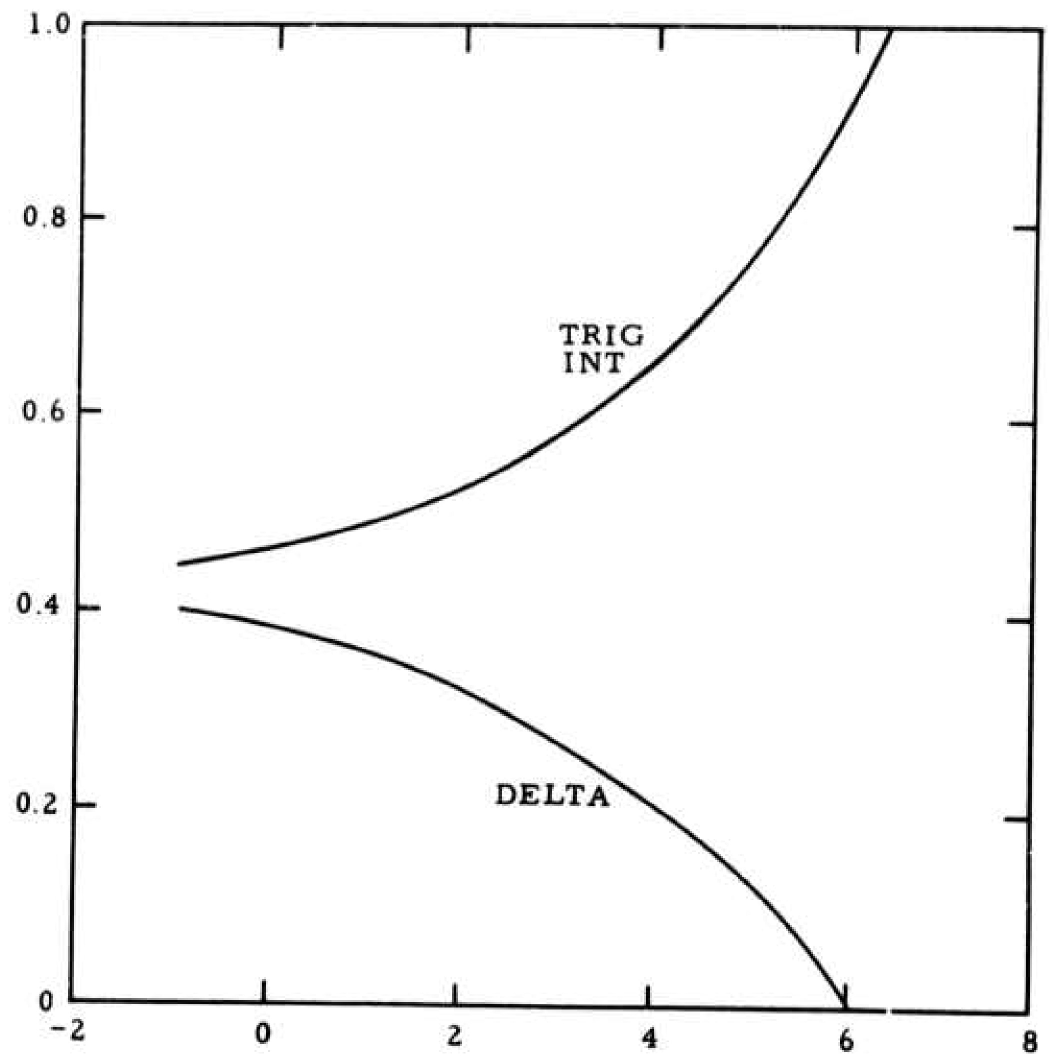

Figure 1. Values of trigonometric integral (theoretical) and delta parameter (empirical) at L = 6.6

The proper generalization of (6) to field lines other than $L = 6.6$ is not evident. However, the trigonometric integral is easily evaluated as a function of L for integer values of m (see Appendix).

In summary, the phase-integral method accurately predicts (within 2%) the spacing $\Delta\omega/2\pi$ between consecutive toroidal eigenfrequencies $\omega_n/2\pi$ of magnetospheric field lines. The eigenfrequencies themselves are accurately given by the formula $\omega_n = (n - \delta)\Delta\omega$, and the value of at $L = 6.6$ is empirically given by (6). The generalization of (6) to other L values is not known, but it is reasonably certain that $\delta = 0$ at all L values for the $m = 6$ plasma-density model.

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APPENDIX: TRIGONOMETRIC INTEGRALS

For the reader's convenience, the integral that appears in (4) is here explicitly evaluated for integer values of m (= 0, 1, 2, . . . , 9). One obtains

$$\int_0^{\sin\Lambda} (1 - x^2)^{3-(m/2)}\, dx =$$

$$(1/35L^3)(16L^3 + 8L^2 + 6L + 5)\sin\Lambda, \quad m = 0;$$

$$(1/48)\left[15\Lambda + L^{-5/2}(15L^2 + 10L + 8)\sin\Lambda\right], \quad m = 1;$$

$$(1/15L^2)(8L^2 + 4L + 1)\sin\Lambda, \quad m = 2;$$

$$(1/8)\left[3\Lambda + L^{-3/2}(3L + 2)\sin\Lambda\right], \quad m = 3;$$

$$(1/3L)(2L + 1)\sin\Lambda, \quad m = 4;$$

$$(1/2)(\Lambda + L^{-1/2}\sin\Lambda), \quad m = 5; \qquad \text{(A1)}$$

$$\sin\Lambda, \quad m = 6; \quad \Lambda, \quad m = 7;$$

$$\tanh^{-1}(\sin\Lambda), \quad m = 8; \quad (L - 1)^{1/2}, \quad m = 9;$$

where $\sin^2\Lambda \equiv 1 - (1/L)$. The results are plotted for $L = 6.6$ in Figure 1.

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NECESSARY CONDITIONS FOR BOUNCE-RESONANT WAVE AMPLIFICATION

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In speculating on the origin of a compressional Pc-4 micropulsation oscillating in the magnetosonic mode at synchronous altitude, Barfield *et al.* (1971) suggested a bounce-resonant diffusion of energy to the wave from the population of ring-current protons. Since the first invariant M is conserved during bounce resonance, a wave-amplifying diffusion of energy would require either an off-equatorial maximum in the mirror-point distribution, or an inward gradient in the L-profile, of particles having in common their value of M. More precisely, the phase-space distribution $\bar{f}$ must be such that either $(\partial\bar{f}/\partial J)_{M,\Phi} > 0$ or $(\partial\bar{f}/\partial\Phi)_{J,M} > 0$ for at least some values of M, J, and Φ (the three adiabatic invariants, all being positive quantities).

It would be useful to have these two amplification criteria expressed in terms of directly measured physical quantities. It is convenient in this context to introduce the kinetic energy (E), the McIlwain parameter (L), the sine of the equatorial pitch angle (y), the cosine of the equatorial pitch angle (x), the scalar momentum (p), and the differential unidirectional flux ($J_\perp$), evaluated at the particle mirror point (*i.e.*, at $B/B_0 = 1/y^2$).

Since $x^2 + y^2 = 1$, it follows from the Jacobian chain rule that

$$\left(\frac{\partial \bar{f}}{\partial J}\right)_{M,\Phi} = -\frac{\partial(M, \bar{f}, \Phi)}{\partial(E, y, L)} \cdot \frac{\partial(E, x, L)}{\partial(M, J, \Phi)} \cdot \frac{x}{y} \tag{1}$$

and

$$\left(\frac{\partial \bar{f}}{\partial \Phi}\right)_{M,J} = -\frac{\partial(M, J, \bar{f})}{\partial(E, y, L)} \cdot \frac{\partial(E, x, L)}{\partial(M, J, \Phi)} \cdot \frac{x}{y}\,, \tag{2}$$

where $\partial(M, J, \Phi)/\partial(E, x, L) = -8\pi \gamma p L^2 a^3 x T(y)$. In this expression (Schulz and Lanzerotti, 1974) the symbol $\underline{a}$ denotes the radius of the earth, γ is the ratio of relativistic mass (m) to rest mass (m_0), and $T(y) \approx 1.3802 - 0.3198\,(y + y^{1/2})$ is the ratio of $p/4mLa$ to the particle bounce frequency. It follows that $-(x/y)[\partial(E,x,L)/\partial(M,J,\Phi)]$ is a positive-definite Jacobian.

Wave amplification therefore requires at least that $\partial(M,\bar{f},\Phi)/\partial(E,y,L) > 0$ or $\partial(M,J,\bar{f})/\partial(E,y,L) > 0$, where $\bar{f} \equiv J_\perp/p^2$ (e.g., Schulz and Lanzerotti, 1974). In other words, wave growth requires that either

$$(\partial \ln J_\perp/\partial \ln y)_{E,L} + 2 - [(\gamma + 1)/\gamma]\,(\partial \ln J_\perp/\partial \ln E)_{y,L} < 0 \tag{3}$$

or

$$4\,(\partial \ln J_\perp/\partial \ln L)_{E,y}\,T(y) - (\partial \ln J_\perp/\partial \ln y)_{E,L}\,Y(y) + [6\,T(y) - Y(y)]\,\{2 - [(\gamma + 1)/\gamma]\,(\partial \ln J_\perp/\partial \ln E)_{y,L}\} < 0 \tag{4}$$

for some values of E, y, and L. The function $Y(y)$ is defined by the relation

$$Y(y) \equiv 2y \int_y^1 u^{-2}\, T(u)\, du. \qquad (5)$$

It varies from $Y(1) = 0$ to $Y(0) = 2T(0)$.

There is no expectation that either condition (3) or condition (4) is sufficient for wave amplification. Even if one or both of these conditions are satisfied for some values of (E, y, L), a properly weighted average over the entire particle distribution is needed for defining the local growth rate (whether positive or negative). Moreover, even if the local growth rate were everywhere positive, spontaneous wave generation would be contingent on adequate internal reflection of the wave energy.

In general the weighted average would involve a linear combination of $(\partial\bar{f}/\partial J)_{M,\Phi}$ and $(\partial\bar{f}/\partial\Phi)_{M,J}$ with the relative weights determined by wave polarization. However, limiting cases can be identified which involve only an unstable pitch-angle diffusion $(\partial\bar{f}/\partial J > 0)$ or an unstable radial diffusion $(\partial\bar{f}/\partial\Phi > 0)$ of the bounce-resonant particles. There is no radial diffusion if the azimuthal component of the wave's electric field is uniform in longitude, and there is no pitch-angle diffusion if the wave is non-compressional.

The case without radial diffusion is interesting in that it represents the geophysical analogue of a two-stream instability if $(\partial \ln J_\perp/\partial \ln E)_{y,L} < 2Y/(Y + 1)$. In this case (e.g., for a particle spectrum that decreases with increasing energy), the necessary condition for wave growth is that $J_\perp$

have a sufficiently pronounced off-equatorial maximum at $B/B_0 = (y^*)^{-2}$. The equatorial distribution of pitch angles in this case has a relative minimum at 90° and absolute maxima at $\sin^{-1} y^*$ and $180^\circ - \sin^{-1} y^*$. An electrostatic instability of this type has been described by Hasegawa and Nishihara (1972). The damping of MHD waves by a distribution failing to satisfy (3) was indicated qualitatively by Roberts and Schulz (1968).

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ELECTROMAGNETIC RADIATION FROM A HELICALLY PHASED PARTICLE BEAM

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The purpose of this note is to examine the concept of a helical beam of particles as an emitter of ULF or VLF waves. Consider a beam of particles, all having speed v and pitch angle α with respect to a uniform magnetic field $\mathbf{B}$. The particles (and also the beam) progress along the field at velocity $v_{\parallel} = v \cos \alpha$, thereby covering a distance $2\pi v_{\parallel}/\Omega$ along $\mathbf{B}$ during each gyration around the central field line. If the particle beam is injected at a fixed phase angle φ_0 from a stationary source at $z = z_0$ along the field, the result is a time-independent helix defined by the equation

$$\varphi(z,t) = (\Omega/v_{\parallel})\, z \;+\; \varphi(0,0). \tag{1}$$

Unless the particle distribution along the helix is modulated in some way (e.g., in particle density), the stationary helix will not constitute an antenna in the usual sense. However, a medium containing such a helically phased velocity distribution can behave as an amplifier of ULF or VLF waves (Sudan and Ott, 1971).

If the current $I(z_0,t)$ of emitted beam particles is modulated in time, the result is a traveling current pattern that radiates in the usual

manner. If the modulation is sinusoidal at $z = z_0$, viz., $I(z_0, t) = I_0 + I_1 \sin(\omega t + \psi_1)$, one obtains

$$I(z,t) = I_0 + I_1 \sin\left[(\omega/v_\parallel) z - \omega t + \psi_1 - (\omega/v_\parallel) z_0\right]. \tag{2}$$

This leads to a current density having an oscillatory component of the form

$$\begin{aligned} \tilde{\underset{\sim}{J}}(\rho, \varphi, z; t) = \\ & (I_1/\rho) \sin\left[(\omega/v_\parallel) z - \omega t + \psi_1 - (\omega/v_\parallel) z_0\right] \\ & \times \delta(\rho - (v_\perp/\Omega))\, \delta(\varphi - (\Omega/v_\parallel) z - \varphi(0,0)) \\ & \times \left[v_\parallel \hat{\underset{\sim}{z}} + v_\perp \hat{\underset{\sim}{\varphi}}\right](1/v) \end{aligned} \tag{3}$$

in cylindrical coordinates (ρ, φ, z). The radiated wave intensity is computed by evaluating the spatial and temporal moments of the waveform with respect to $\tilde{\underset{\sim}{J}}(\rho, \varphi, z; t)$.

If the emitted beam is not modulated, a wave source can still be created by giving the emitter a constant velocity $\dot{z}_0$ along $\underset{\sim}{B}$, or by allowing the gyrophase angle φ_0 of the beam to vary at a constant rate $\dot{\varphi}_0$ at the time of injection, or both. In this case one obtains a non-stationary helix defined by the equation

$$\varphi(z,t) = \frac{\Omega - \dot{\varphi}_0}{v_\parallel - \dot{z}_0}\, z + \frac{\dot{\varphi}_0 v_\parallel - \dot{z}_0 \Omega}{v_\parallel - \dot{z}_0}\, t + \varphi(0,0), \tag{4}$$

which constitutes a wave source unless $\Omega = \dot{\varphi}_0$ or $\dot{\varphi}_0 v_\parallel = \dot{z}_0 \Omega$. In the latter degenerate case one obtains only a stationary helix analogous to (1), and wave generation is possible only through instabilities of the beam-plasma system.

The evaluation of growth rates for such plasma instabilities can be greatly simplified by making the problem homogeneous with respect to ρ. This situation corresponds to an infinite array of phased beam emitters, distributed uniformly in a plane perpendicular to $\underset{\sim}{B}$. The experimental unreality of such a configuration need not be of concern if the waves in question are well guided along $\underset{\sim}{B}$, i.e., if $|\hat{\underset{\sim}{B}} \times \underset{\sim}{\nabla}_k \omega| \ll |\hat{\underset{\sim}{B}} \cdot \underset{\sim}{\nabla}_k \omega|$ throughout the interesting region of $\underset{\sim}{k}$ space. In this case, or if the waves are ducted along $\underset{\sim}{B}$ by field-aligned plasma inhomogeneities, the approximation of a transversely homogeneous plasma will normally yield about the same growth rates and wave amplitudes as the fundamentally more valid wave-packet formulation.

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COMPUTER SIMULATION OF AN ION BEAM ANTENNA

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ABSTRACT

The feasibility of using a modulated beam of ions as an antenna in the magnetosphere is discussed briefly. A computer simulation of an ion beam in a plasma is presented in which the motion of the ions is prescribed while the electrons in the surrounding plasma are represented by concentric charged shells, each of which moves in the self-consistent field set up by the beam and the other shells. By following the motion of the shells on a computer, an attempt is made to estimate the sheath current that is induced in the plasma by the beam.

Introduction

It is known that ULF and VLF waves can propagate readily in the upper ionosphere because of an amplifying mechanism based on an electromagnetic cyclotron instability that is now well understood.[1] However, for communication purposes, it is not clear how such waves can be most effectively generated. One possible scheme that has been suggested is to create an antenna consisting of a beam of ions launched from a satellite.[2]

Radiation from an ion beam can be obtained, at least in principle, by any one of the following methods: modulation of (a) ion energy, (b) ion linear density in the beam, (c) drift velocity of the electrons along the beam, and (d) electron linear density in the beam. Alternative (c) is probably the most readily achieved. In alternative (c) an ion-beam source maintains a constant ion current. The beam is space-charge neutralized at all times by electrons injected into the beam. The voltage required to modulate the electron current is then much smaller (by the square root of the mass ratio) than that which would be required for the ions.

A possible configuration for the ion-beam source would consist of a Penning discharge and extraction of the ions through acceleration/deceleration grids, similar to the electron-bombardment ion source developed by NASA for spacecraft electric propulsion. However, in contrast to these sources, a much smaller beam divergence is needed in the present

application. This can be achieved by relaxing the requirement of high current density (and therefore high efficiency) imposed on electric propulsion ion-sources. Electrons are injected into the ion beam in such a way that the electron current can be modulated, although the average current is equal to the ion current. Typical requirements might be a 50 mA ion current of 5 keV Hg ions and a 20% modulation amplitude of the neutralizing electron current.

An understanding of the dynamics, sheathing, stability, and radiation from such a beam through theoretical efforts and experimental measurements is required to verify that a beam is a viable complement or alternative to a conventional mechanical antenna.

To illustrate the typical range of the physical parameters involved, we have considered two cases--an ion beam antenna on a spacecraft (a) inside the plasmapause, for a magnetic shell parameter $L = 3.0$, and (b) outside the plasmapause, at $L = 5.0$. The results are summarized in Table I.

The effective length of the ion beam antenna has been computed by estimating the divergence of the beam and by assuming that the beam effectively terminates at the point where the beam particle density has become equal to the ambient charged-particle density. This omits any consideration of beam/plasma instabilities that, if present, may result in an effective length smaller than the one assumed here. Details of the calculation of beam divergence are summarized in Appendix I.

As is seen from the Table, the cyclotron radius of the beam ions is much larger than the effective antenna length; therefore, the effect of

TABLE I

Typical Parameter Values for Ion Beam Antenna

Assumed ion source: 5 keV Hg-ions, 50 mA ion-current

	Inside Plasma pause L = 3.0, equatorial	Outside plasma pause L = 5.0, equatorial
ambient electron density (daytime)	$n = 2 \cdot 10^{3}\ cm^{-3}$	$50\ cm^{-3}$
magnetic field	$H = 10^{-2}$ Gauss	$2 \cdot 10^{-3}$ Gauss
cyclotron frequency of beam ions (Hg)	$\omega_{c+} = 0.5\ sec^{-1}$	$0.1\ sec^{-1}$
cyclotron radius of beam ions (Hg)	$r_{c+} = 120$ km	600 km
effective beam length	$\ell = 1.0$ km	6.0 km
ambient plasma frequency	$\omega_p = 2.5 \cdot 10^{6}\ sec^{-1}$	$4 \cdot 10^{5}\ sec^{-1}$
electron cyclotron frequency	$\omega_{c-} = 2 \cdot 10^{5}\ sec^{-1}$	$4 \cdot 10^{4}\ sec^{-1}$
ion cyclotron frequency (protons)	$\omega_{cH} = 100\ sec^{-1}$	$25\ sec^{-1}$
signal frequency	$\omega/2\pi = 1000$ Hz	500 Hz
phase index of refraction for longitudinal propagation	$n_{\parallel} = 70$	30
wave length in plasma	$\lambda = 4$ km	15 km
ratio of antenna length to 1/4 wave length	$\frac{4\ell}{\lambda} = 0.8$	1.3

the ambient magnetic field on the beam ions can be neglected.

The phase refractive index for longitudinal propagation in the ambient plasma has been computed on the basis of the cold-plasma approximation. From this, and from the assumed signal frequency, the wavelength in the plasma can be estimated and compared with the effective antenna length. For the stated values of the parameters, the effective antenna length is found to be roughly equal to one quarter wavelength, which satisfies one of the essential conditions for efficient radiation.

Apart from spreading and possible instability, the efficiency of the beam as an antenna is affected by the electric-current distribution that it induces in the ambient plasma. In this report we shall concentrate on this question only and try to estimate the sheath current around a perfectly collimated beam of infinite length.

In principle a perfectly neutralized stationary beam could exist in a cold plasma without any sheath current being formed. However, modulation of the beam current will cause the electrons to experience an induction force due to the changing magnetic field, and hence to move in such a way as to oppose the modulation. On the other hand, charge modulation will cause the electrons to move in a complicated way through the magnetic field in response to the electrostatic forces. We consider this case first. Since the response of the electrons will be very fast compared to the modulation period, we are probably then justified in considering a stationary beam that is incompletely neutralized. Then the electric and magnetic fields surrounding it will cause a plasma electron

released in its vicinity to move in such a way as to produce a current opposing the beam current. Trajectories of single electrons released from rest in the neighborhood of a charged beam are shown in Figure 1. The $\underset{\sim}{V} \times \underset{\sim}{B}$ force which they experience causes them to move in the direction of the beam ions thus shielding the beam current. The more realistic picture of a beam immersed in a plasma is, of course, much more complicated than this single-particle picture suggests, since then each electron must move in the self-consistent field set up by all the other electrons as well as the beam. To treat this problem in general even in a cold plasma would require integrating a set of partial differential equations in 4 dimensions. Instead, to keep the computation tractable, we choose a simplified model in which the beam consists of a circular cylinder of charge moving axially, the earth's magnetic field is excluded, and the plasma is represented by a series of concentric cylindrical shells carrying negative charge, which are free to move both axially and radially through a uniform background of fixed positive charge. Later we shall briefly consider the effect of current modulation of an uncharged beam.

Each electron shell is then acted on by 4 sets of forces, namely those due to the beam, the other shells, the ion background and, finally, its own self force. Initially, under the influence of the beam's electrostatic attraction, the shells fall inwards, that is their radii diminish. We assume that the shells can pass freely through the beam and through each other. As time progresses the inner shells collapse down to the point that the electrostatic repulsion due to their self force causes them

to reverse their motion and move away from the beam; note that this behavior is different from that of the individual electrons in that the latter pass through the axis of the beam. The radial motion of the shells, either inwards or outwards, in the presence of the beam's magnetic field induces an axial motion of the shells and hence an axial current surrounding the beam. The current in any particular shell is always such as to oppose the beam's current initially when the shell is moving in, but after the shell bounces and moves outward passing through other incoming shells on its way, it may acquire a sufficiently large positive radial motion that its axial motion is reversed, and it then carries a beam reinforcing current. Eventually the outward radial motion is itself arrested and the shell falls in again. This complicated turbulent motion is confined to the inner shells. The outer shells see the beam's electrostatic field to be very quickly neutralized, and as a result they move very little. We define the sheath current to be the total integrated current in all the shells.

Although this model is clearly too complicated to treat except numerically, it already represents an extreme simplification of the actual situation. One of its most serious shortcomings is probably the absence of an external magnetic field, since in fact the earth's field is not expected to be small compared to the beam field. However, this assumption may be a conservative one in the sense that an external field would probably reduce the shielding currents induced in the plasma. If this is true, then the currents calculated on this model should be pessimistically large. Another defect is the absence of any axial variation. We discuss this point later.

The Model

We now define the model explicitly as follows. The beam consists of a circular cylinder of radius $\underline{a}$ carrying a uniform distribution of positive charge of density Zen_+ moving axially with a velocity w_+. The beam is immersed in a plasma consisting of a uniform ion background of charge density en and infinite mass together with a series of shells concentric with the beam and located initially at radii $r_i = i\Delta r$ and carrying charge $q_i = -en\, 2\pi r_i\, \Delta r$ and mass $M_i = mn\, 2\pi r_i\, \Delta r$, where the subscript refers to the i^{th} shell. Using z for the axial coordinate, we can write the radial equation of motion of the ith shell as follows

$$\ddot{r}_i = -\frac{e}{m}\left[(E_r)_i - \dot{z}_i B_i\right] \tag{1}$$

where the radial electric field

$$(E_r)_i = (E_r)_i^{(b)} + (E_r)_i^{(i)} + (E_r)_i^{(e)} \tag{2}$$

is made up of the electric fields due to the beam, the ions and the electrons.

For the beam we have

$$(E_r)_i^{(b)} = \frac{n_+ ea}{2}\; f(r_i/a) \tag{3}$$

where $f(x) = x\,H(1 - x) + (1/x)\,H(x - 1)$, $H(x)$ being the step function. For the ion background

$$(E_r)_i^{(i)} = \frac{n_+ e}{2\epsilon_o} r_i \tag{4}$$

and for the electron shells

$$(E_r)_i^{(e)} = -\frac{n e}{\epsilon_o r_i} \overline{\Delta r}^2 \underline{\underline{\sum}}^{i} (q_j) \tag{5}$$

where

$$\underline{\underline{\sum}}^{i} (q_j) = \sum_{\{j:\ r_j < r_i\}} q_j + \frac{1}{2} q_i \tag{6}$$

We put $q_j = (n_-/n)j$ for $j \le N_B$ and $q_j = j$ for $j > N_B$ where N_B is the number of shells inside the beam, $N_B = a/\Delta r$. Taking $n_- > n$ allows for neutralizing electrons to be ejected with the beam. The form of $\underline{\underline{\sum}}^{i}(q_j)$ expresses the fact that the electric field at the i^{th} shell is made up of a sum of contributions from just those shells that lie beneath it plus a part coming from the electrostatic forces of the shell on itself.

Similarly, the axial equation of motion has the form

$$\ddot{z}_i = -\frac{e}{m}\left[(E_z)_i + \dot{r}_i B_i\right] \tag{7}$$

Here B_i represents the azimuthal magnetic field at the i^{th} shell due to the beam current and the electron-shell currents.

$$B_i = \mu_o nea \left[\frac{1}{2} \frac{n_+}{n} w_+ f(\frac{r_i}{a}) - \frac{\overline{\Delta r}^2}{r_i a} \sum^i (q_j \dot{z}_j)\right] \tag{8}$$

Finally, the axial electric field is related to the magnetic field by Faraday's law

$$\frac{d}{dr}(E_z)_i = \dot{B}_i \tag{9}$$

If one attempts to solve these equations in a medium of infinite radial extent, one finds an unbounded time varying flux $\int B\, dr$ due to the shells, which in turn leads to infinite induction forces through Eqs. (9) and (7). To get around this difficulty we assume that the beam current is returned at a large but finite distance r_∞. Fortunately, the results turn out to be rather insensitive to the value chosen for r_∞. In practice, the beam length is also finite, and one would expect r_∞ to be comparable to it.

To eliminate radial derivatives it is convenient to introduce a new dependent variable

$$\phi_i = \int_{r_i}^{r_\infty} B(r)\, dr \tag{10}$$

representing the flux per unit length of beam between r_i and r_∞. Substituting (8) into (10), we find for ϕ_i

$$\phi_i = \mu_o \, nea^2 \, c \left[\frac{1}{2} \, \frac{n_+}{n} \, \frac{w_+}{c} \, g\left(\frac{r_i}{a}\right) - \left(\frac{\Delta r}{a}\right)^2 \sum_{j=1}^{\infty} q_j \, \frac{\dot{z}_j}{c} \ln \frac{r_\infty}{r_{ij}} \right] \tag{11}$$

Here $g(r/a) = \int_{r_i}^{r_\infty} f(r/a) \, dr/a$ and $r_{ij} = \max(r_i, \; r_j)$. Note that in this expression the summation is over all the shells. We next differentiate (11) to find $\dot{\phi}_i$, the time derivative of ϕ_i

$$\dot{\phi}_i = \mu_o \, nea^2 \, c \left\{ \frac{1}{2} \left(\overline{\frac{n_+}{n} \, \frac{w_+}{c}}^{\,\bullet} \right) g\left(\frac{r_i}{a}\right) - \left(\frac{\Delta r}{a}\right)^2 \left[\sum_{j=1}^{\infty} q_j \, \frac{\ddot{z}_j}{c} \ln \frac{r_\infty}{r_{ij}} - \overline{\overline{\sum_i}} \left(q_j \, \frac{\dot{r}_j}{r_j} \, \frac{\dot{z}_j}{c} \right) \right] \right\} \tag{12}$$

Here

$$\overline{\overline{\sum_i}} (q_j) = \sum_{\{j: \, r_j > r_i\}} q_j \quad + \frac{1}{2} \, q_i \tag{13}$$

enters because only the radial motion of shells __above__ the i^{th} contributes to the induction. In terms of $\dot{\phi}_i$ Eq. (7) takes the form

$$\ddot{z}_i = - \frac{e}{m} (\dot{\phi}_i - \dot{r}_i B_i) \tag{14}$$

When we substitute for $\dot{\phi}_i$ and B_i from Eqs. (12) and (8) into Eq. (14) we find the summations involving z_i can be combined, and we have

$$\ddot{z}_i = \frac{e^2 n}{m \epsilon_o} \frac{a^2}{c} \left\{ \frac{1}{2} \dot{\overline{\frac{n_+}{n} \frac{w_+}{c}}} g_i - \frac{1}{2} \frac{n_+}{n} \frac{w_+}{c} \frac{r_i}{a} f_i \right.$$

$$\left. - \left(\frac{\Delta r}{a}\right)^2 \sum_{j=1}^{\infty} \left[q_j \frac{\ddot{z}_j}{c} \ln \frac{r_\infty}{r_{ij}} - j \frac{\dot{r}_{ij}}{r_{ij}} \frac{\dot{z}_j}{c} \right] \right\} \tag{15}$$

Here $\dot{r}_{ij} = \dot{r}_i H(r_i - r_j) + \dot{r}_j H(r_j - r_i)$.

We now simplify the notation by introducing dimensionless variables

$$\begin{aligned} r &= a\,\bar{r} & \qquad n_\pm/n &= N_\pm \\ t &= \omega_p^{-1}\,\bar{t} & w_\pm/c &= W_\pm \\ \dot{r} &= \omega_p\, a\,\bar{u} & \Delta r/a &= \delta \\ \dot{z} &= \nu\,\omega_p\, a\,\bar{w} & & \end{aligned}$$

where $\omega_p^2 = e^2 n/\epsilon_o m$ is the square of the electron plasma frequency and $\nu = \omega_p a/c$. Thus lengths are measured in units of the beam radius and times in units of the plasma period. Notice that the radial and axial velocity components are scaled differently. In terms of these quantities (and dropping the bars) we have

$$\dot{r}_i = u_i \tag{16}$$

$$\dot{u}_i = -\frac{1}{2}\; N_+ f_i - \frac{1}{2}\; r_i + (\delta / r_i) \sum^{i} (q_j) \tag{17}$$

$$\dot{w}_i + \epsilon \sum_j A_{ij}\; \dot{w}_j = F_i + \epsilon \sum_j C_{ij}\; w_j \tag{18}$$

where

$$A_{ij} = q_j \ln (r_\infty / r_{ij})$$

$$C_{ij} = q_j\; u_{ij} / r_{ij}$$

$$F_i = \dot{\beta}\; g_i - \beta\, u_i f_i$$

$$\beta = \frac{1}{2}\; W_+\; N_+$$

$$\epsilon = (\nu\; \delta)^2$$

and the dot now represents differentiation with respect to $\omega_p t$. The $r_i B_i$ term occurring in (7) is excluded in (17) by the assumptions $\nu^2\; W_+ \ll 1$ and $\nu^4 \ll 1$.

Equations (16) - (18) represent a set of $3N_S$ coupled differential equations, where N_S is the number of shells which we should like to take as large as possible. This task is clearly impractical, even on a high-speed computer. Note that in matrix notation (18) reads

$$(I + \epsilon A)\dot{w} = F + \epsilon C w \tag{19}$$

where I, A and C are square matrices with I the unit matrix and F and w column symbols. We shall now make the assumption that $\epsilon \, \|A\| \ll 1$ and $\epsilon^2 \ll 1$. (In the numerical work we shall take $\delta = \Delta r/a = 0.2$ and $\nu = 0.01$, which makes $\epsilon = 4 \times 10^{-6}$. This value of ν corresponds for example in the case of a 10 cm beam to $\omega_p = 3.10^7 \text{ sec}^{-1}$.) Finally, the elements of A are bounded roughly by $N_S \ln N_S$. Accordingly, we premultiply Eq. (19) by $(I + \epsilon A)^{-1} \approx I - \epsilon A$ to obtain

$$w_i = F_i + \epsilon \sum_j (C_{ij} w_j - A_{ij} F_j) \tag{20}$$

Unlike (19), these equations are now uncoupled. We shall be particularly interested in calculating the total current in the plasma and relating it to the current in the beam. In particular we shall attempt to compute the following quantity

$$\mathscr{R} = \frac{J_S - J_E}{J_I - J_E} \tag{21}$$

where J_I and J_E are the total ion and electron currents in the beam initially and J_S is the total current carried by all the shells and hence includes J_E. With this normalization $\mathscr{R} = 0$ corresponds to no sheath current induced in the plasma and $\mathscr{R} = 1$ corresponds to perfect shielding.

We have

$$J_I = \pi a^2 e\, n_+ w_+$$

$$J_E = 2\pi \overline{\Delta r}^2 e\, n \sum_{i=1}^{N_B} q_i w_i$$

$$J_S = 2\pi \overline{\Delta r}^2 e\, n \sum_{i=1}^{N_S} q_i w_i \tag{22}$$

In terms of the dimensionless quantities $\overline{J}_I = N_+ V_+$, $\overline{J}_E = N_- V_-$ and $\overline{J}_S = 2\epsilon \sum^{i} (q_j\, \overline{w}_j)$ we can write

$$\mathscr{R} = \frac{\overline{J}_S - \overline{J}_E}{\overline{J}_I - \overline{J}_E} \tag{23}$$

Numerical Computation

Equations (16), (17) and (18) have been programmed for numerical computation. In all cases the number of shells N_S was arbitrarily set at 200 with $N_B = 5$ shells inside the beam so $\Delta r/a = 0.2$. The results of a particular run are shown in Figures 2 - 4. For this case the beam was assumed to be steady ($\dot{\beta} = 0$) and the beam ions to be traveling at 1% the speed of light ($W_+ = .01$), corresponding for example to 50 kV

protons. The beam is unneutralized with the ion density equal to 4 times the background density ($N_+ = 4$, $N_- = 1$). Figure 2 is a computer plot of the radial motion of the first 50 shells as a function of time. Figure 2b is a similar plot of every 4th shell from the 4th to the 200th. It is clear that bouncing is confined to the inner shells while the outer shells merely pulsate at roughly the plasma frequency. The current ratio $\mathscr{R}$ is plotted in Figure 3. Here the five curves represent the total electron current (normalized to the beam current) under fixed radial distances, namely 8, 16, 24, 32 and 40 beam radii. These curves are labeled 1 to 5, respectively. The fact that they are quite close together indicates that the sheath current is largely confined to the vicinity of the beam. It exhibits a pulsation in step with the motion of the shells, but after a few plasma periods phase mixing sets in and the sheath current is reduced. In Figs. 2 and 3 the beam current has been made to return at the initial position of the 200th shell, i.e., at 40 beam radii, $r_\infty = 40$. Figure 4 is similar to Fig. 3 but with $r_\infty = 400$, the region between $r = 40$ and $r = 400$ being vacuum. The total sheath current and its distribution is seen to be only slightly affected by the position of the return current.

The beam currents considered so far are, of course, much smaller than those that would be of practical interest, but they reveal the qualitative response of the plasma. We next raise the beam current by increasing N_+ to 42. At the same time we maintain partial neutralization by taking $N_- = 35$. (Note that, because of the discretisation error, perfect neutralization would require taking $N_- = 1 + [N_B/(N_B+1)]N_+$, which is 36 in our case). Again the sheath current shown in Fig. 5 is qualitatively similar to that in the previous case except that the oscillation is now at

a higher frequency corresponding to the fact that the density within the beam is higher. Also the total sheath current is now even smaller than before as a result of the smaller excess charge on the beam.

So far the electron shells within the beam have been assumed to be initially at rest, $W_- = 0$. Of more practical interest is the case where they are given an initial velocity $W_- > 0$ and at the same time a high density $N_- \gg 1$. Then it is found that in the absence of perfect neutralization the beam is effectively unstable since these inner shells promptly leave the beam taking a large part of the current with them. Of course these shells are just as promptly replaced by others from the surrounding plasma to preserve neutrality. We conclude that in any practical situation where a portion of the beam current (perhaps a major portion) will be carried by the electrons it is important that as high a degree of neutrality as possible exist in the beam initially as it is ejected to prevent its being broken up by electrostatic forces. In any cases sheath currents due to charge effects of the kind we have considered here are quite negligible.

Up to now we have always assumed the beam current to be steady, i.e., we have taken $\dot{\beta} = 0$ in Eq. (18). In practice we are interested in the case where the beam current is modulated, albeit at a very slow rate compared to the plasma frequency. However, because of its mathematical simplicity we look next at the opposite extreme, namely a beam that is turned on abruptly at time $t = 0$. Then β in Eq. (18) becomes a step function and $\dot{\beta}$ a delta function. The axial velocity distribution at time $t = 0^+$ becomes

$$w_i = W_+ N_+ (g_i - \epsilon \sum_j A_{ij} g_j) \tag{24}$$

The resulting initial current distribution throughout the plasma is shown in Fig. 6. It is clearly no longer confined to a sheath in the vicinity of the beam, and it now depends strongly on r_∞, the position of the return currents. Furthermore, it is a significant fraction of the beam current. These curves remain effectively constant in time since current arising from radial shell motion superimposes only a small ripple.

In practice the beam would be turned on very slowly with a build-up time long compared to a plasma period. However, such a slow build-up is not expected to change the final results significantly on the basis of the present model. Rather, these results point up an inadequacy of the model arising from the neglect of an axial variation and of displacement current. Both of these effects are taken into account in an accompanying paper[3] which includes a full-wave treatment of the antenna, but at the cost of linearizing the problem. If the perturbation of the plasma by the beam is in fact small, as we believe it is, then the sheath current should be adequately accounted for in the near field of the solution given in Ref. 3.

Acknowledgement

The authors have benefited from helpful discussions with Professor J. M. Cornwall and Dr. R. H. Huddlestone.

APPENDIX I: Ion Beam Spread

The effective length of the ion beam as an antenna is limited by a number of effects. Collisions with the ambient plasma and finite ion Larmor radius effects are negligible under the assumed conditions, but effects such as beam spreading due to imperfections in the ion-optics of the source, ion and electron temperature effects, beam-plasma instabilities, and electrostatic repulsion in the beam (for incompletely space-charge neutralized beams) all may play a role.

We briefly consider here the effect of the ion temperature on the beam spreading. The best collimation (beam half-angle β) that can be achieved is of the order of

$$\beta \approx \frac{w_{th+}}{w_+} \approx \left(\frac{kT_+}{eV_+}\right)^{1/2} \qquad (1)$$

where w_+ and w_{th+} are the axial and thermal velocities respectively of the beam ions, V_+ the accelerating potential of the ion source, and T_+ the ion temperature in the source. T_+ depends on the details of the ion source design, but it can be estimated from the sheath potential. Based on current ion-source technology, $kT_+ = 5$ eV is assumed.

The ion beam current is $I_+ = e\, n_+(z)\, w_+\, A(z)$, where $n_+(z)$ is the beam ion density as a function of distance z from the source, $w_+ = (2eV_+/m_+)^{1/2}$ the ion axial velocity and A(z) the cross-sectional area of the beam. For a circular aperture, at sufficient distance from the source (A >> area of aperture),

$$A(z) = \pi(\beta z)^2 \qquad (2)$$

We take as the effective termination of the beam the point where

$$n_+(\ell) = n \tag{3}$$

(n = electron/ion density of the ambient plasma). This condition leads to the expression

$$\ell = I_+^{1/2}(\pi en)^{-1/2}\left(\frac{2e}{m_+}\; V_+\right)^{-1/4} \beta^{-1} \tag{4}$$

for the effective beam length ℓ. Numerical estimates for ℓ are contained in Table I.

It is also of interest to note that the beam length depends only weakly on the accelerating potential of the ion source ($\ell \sim V_+^{1/4}$ for constant T_+, i.e., $\beta \sim V_+^{-1/2}$) and also only weakly on the ion mass ($\ell \sim m_+^{1/4}$).

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FIGURE CAPTIONS

Fig. 1 Single-particle trajectories for electrons released from rest at 2 and 5 beam radii for beams of charge-density ratio $N_+ = 4$ and 42.

Fig. 2 Shell motion surrounding a beam of beam/background charge-density ratio $N_+ = 4$. a) first 50 shells; b) every 4th shell from 4 to 200. Initially the shells are evenly spaced with the beam extending over the first 5.

Fig. 3 Shell current normalized to beam current for the case $N_+ = 4$, $N_- = 1$, $W_+ = .01$, $W_- = 0$, $\nu = .01$, $r_\infty = 40$. Curves 1 to 5 refer to total shell current under 8, 16, 24, 32, and 40 beam radii, respectively.

Fig. 4 Similar to Fig. 3, but for $r_\infty = 400$.

Fig. 5 Shell current normalized to beam current for the case $N_+ = 42$, $N_- = 35$, $W_+ = .01$, $W_- = 0$, $\nu = .01$, $r_\infty = 400$.

Fig. 6 Initial shell-to-beam current ratio for a beam turned on abruptly at time $t = 0$ for different values of r_∞. Other parameters are the same as in Fig. 4.

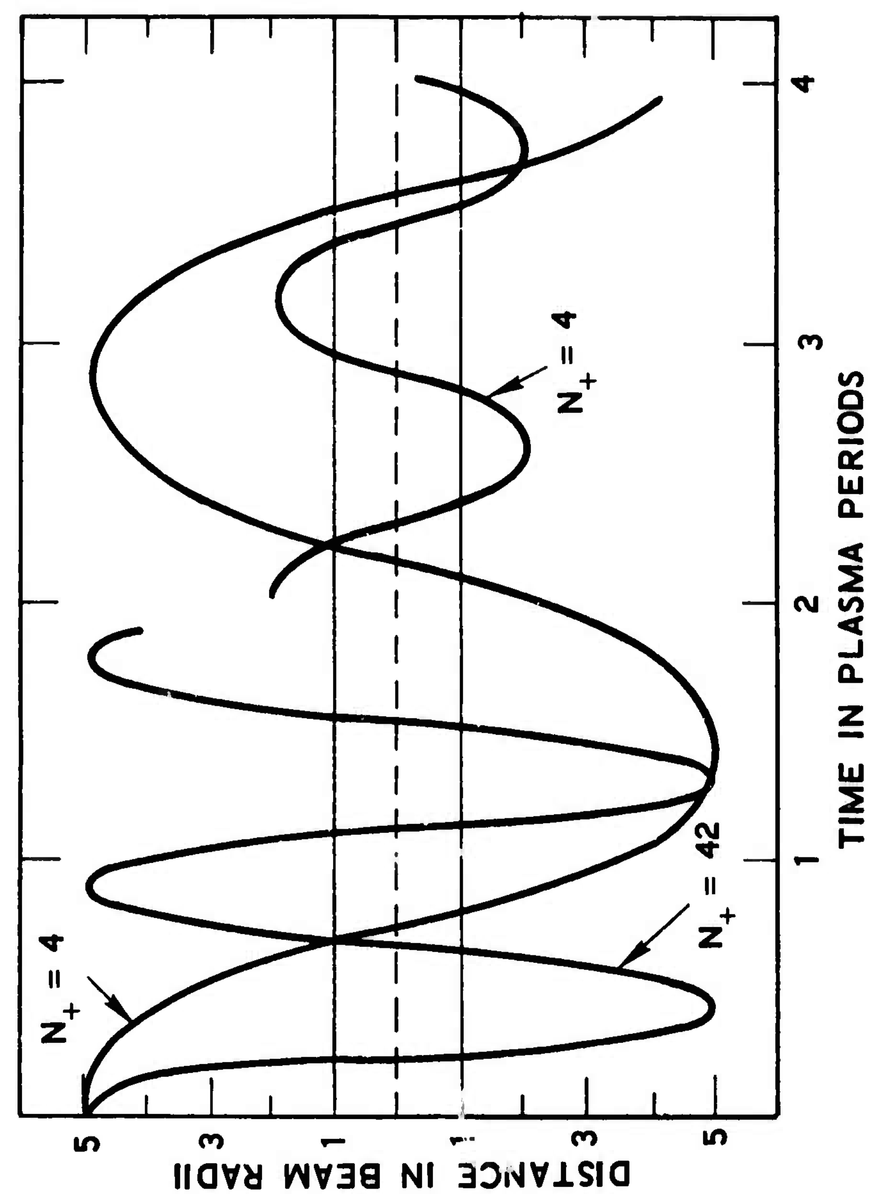

DISTANCE IN BEAM RADII
5
3
1
1
3
5
TIME IN PLASMA PERIODS
1
2
3
4
$N_+ = 4$
$N_+ = 42$
$N_+ = 4$

Figure 1

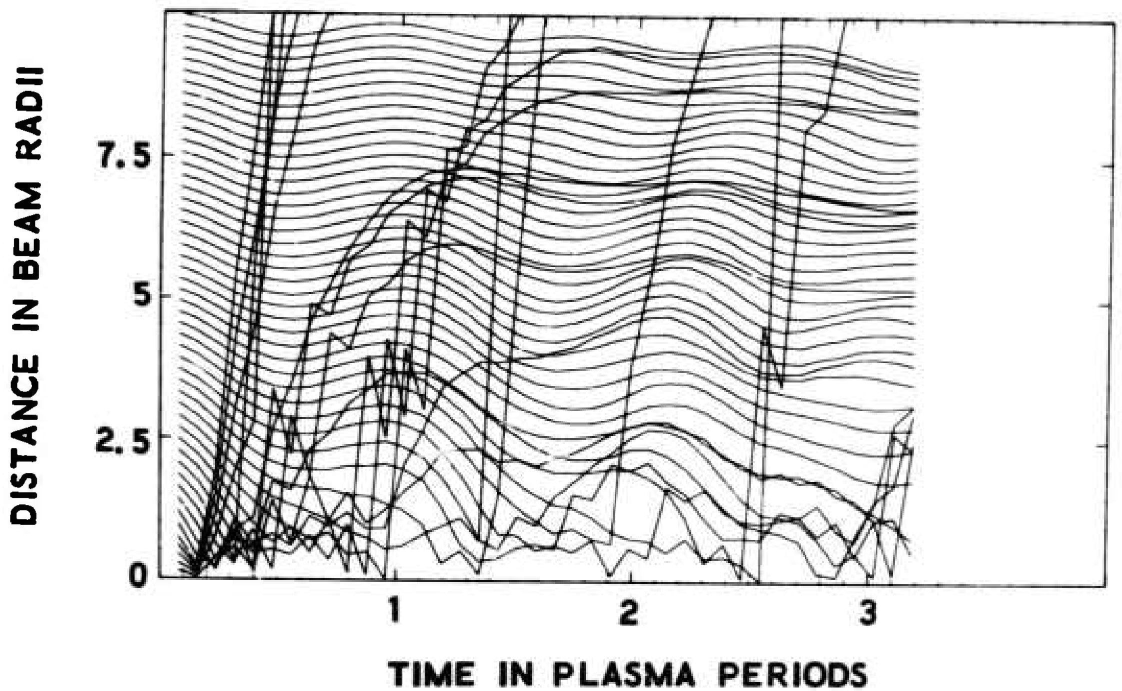


Figure 2a

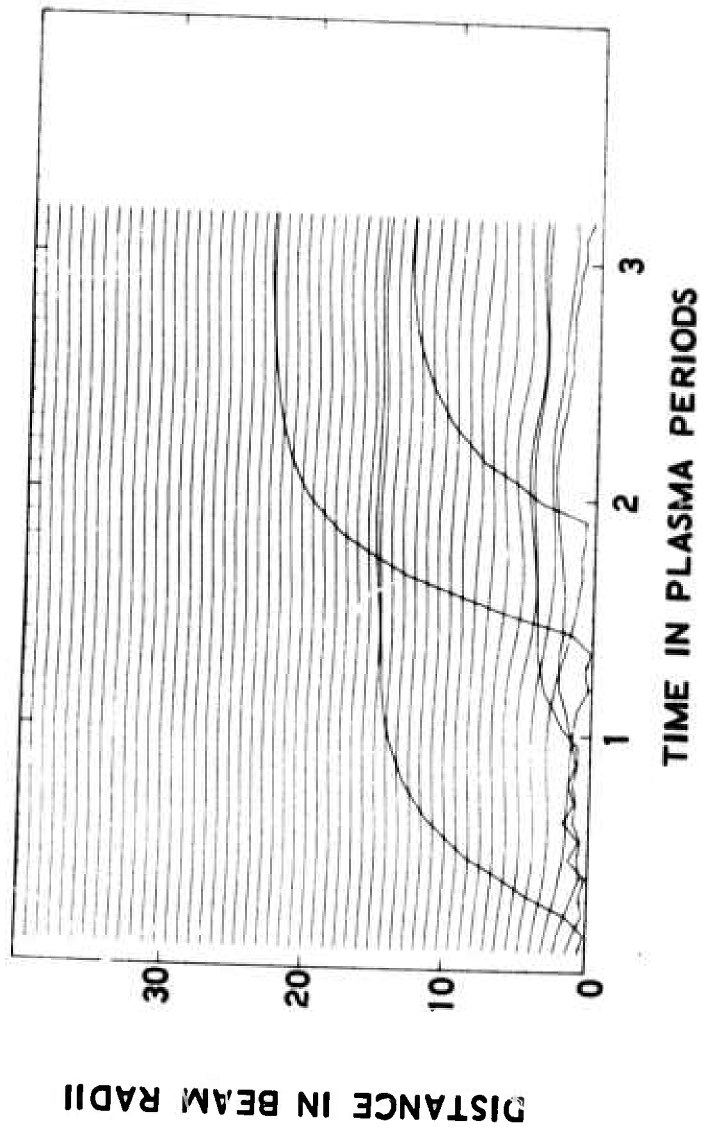
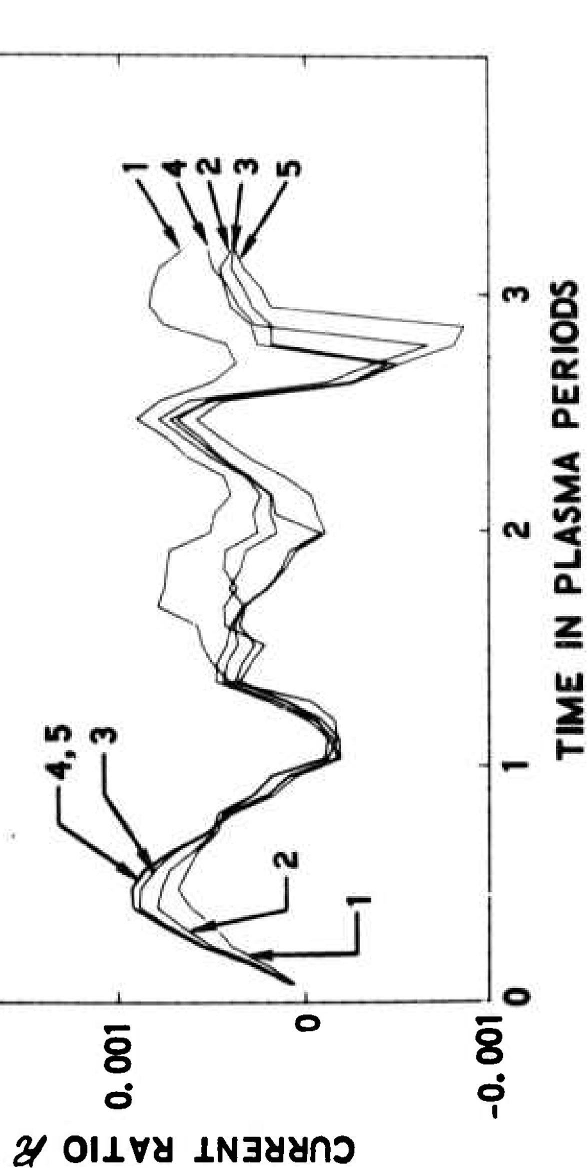


Figure 2b



CURRENT RATIO $\mathcal{R}$
0.001
0
-0.001
0
1
2
3
TIME IN PLASMA PERIODS
1
2
3
4,5
1
4
2
3
5

Figure 3

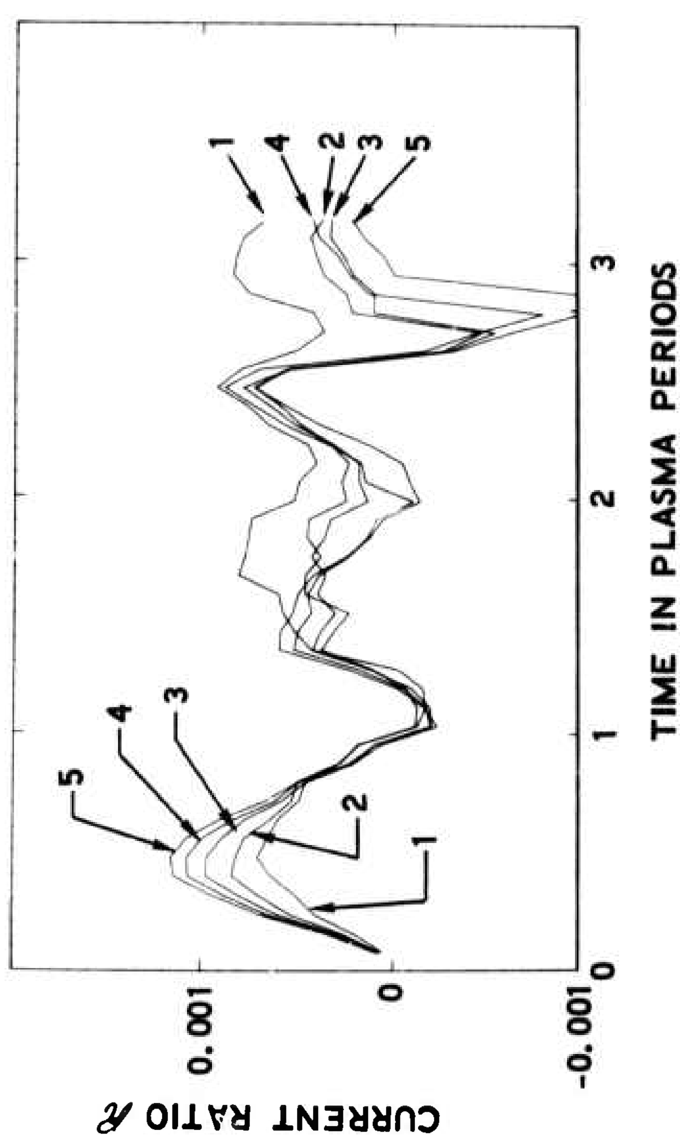

CURRENT RATIO ℛ
0.001
0
-0.001
0
1
2
3
TIME IN PLASMA PERIODS
1
2
3
4
5

Figure 4

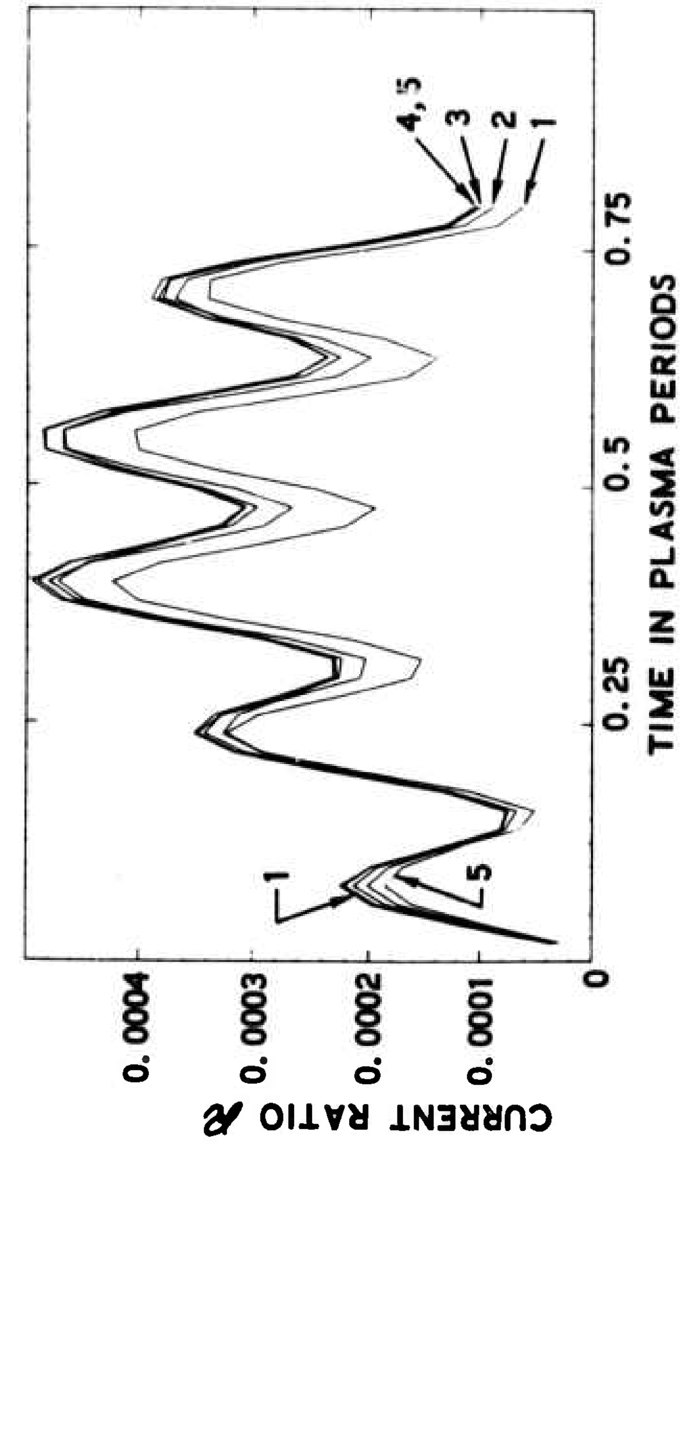

CURRENT RATIO R
0.0004
0.0003
0.0002
0.0001
0
0.25
0.5
0.75
TIME IN PLASMA PERIODS
1
5
4,5
3
2
1

Figure 5

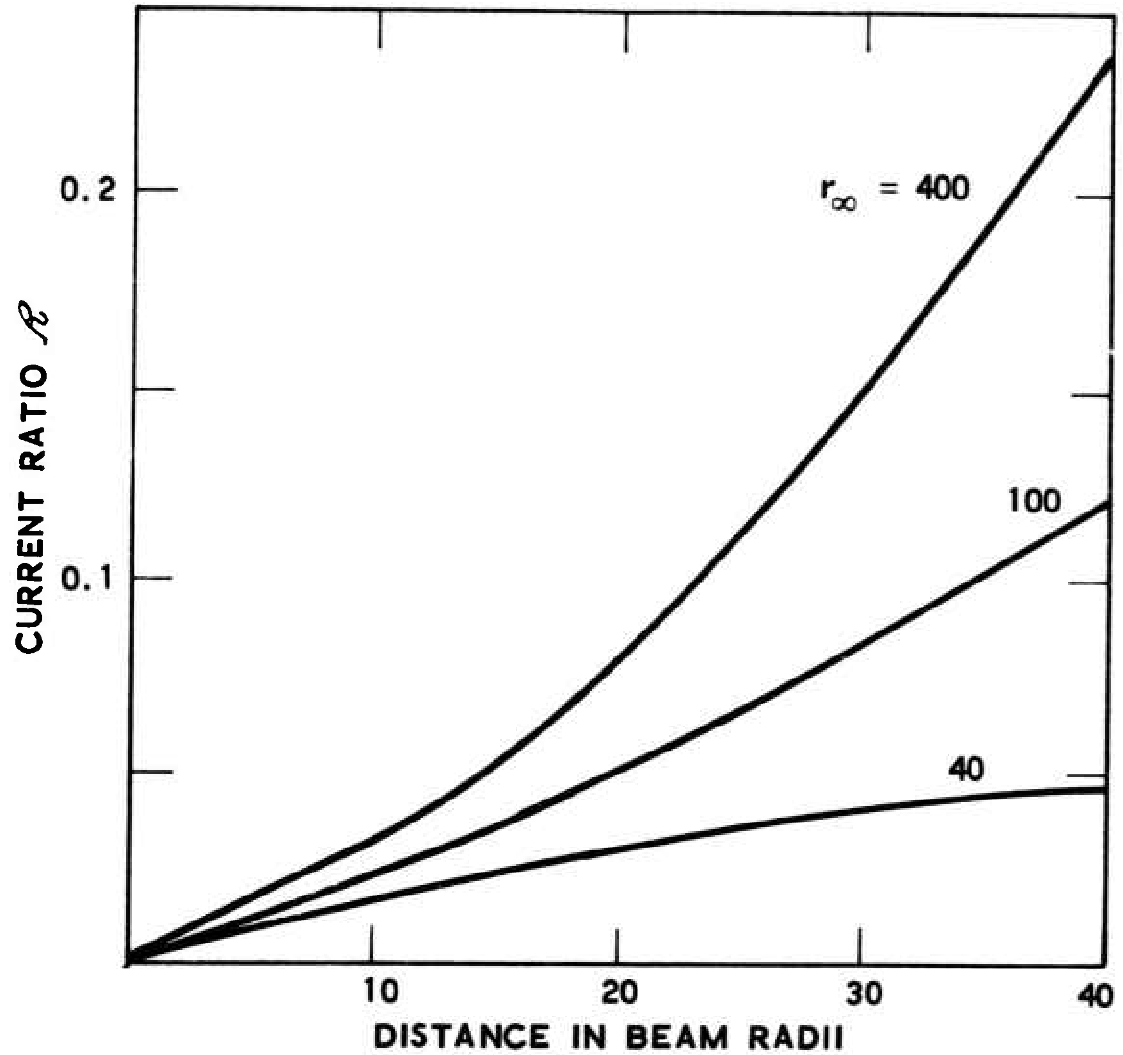

CURRENT RATIO $\mathscr{R}$
0.2
0.1
r_∞ = 400
100
40
10
20
30
40
DISTANCE IN BEAM RADII

Figure 6

RADIATION FROM A LINE SOURCE CARRYING A TRAVELING WAVE IN A MAGNETOPLASMA

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Many papers have been written on radiation in the ionosphere from dipoles or finite antennas carrying a prescribed standing-wave current distribution.[1] For some applications it is of interest to calculate the radiation from a line antenna of infinite length carrying a current distribution that is traveling along it at a prescribed speed. A possible application would be to an antenna in the form of a modulated beam of charged particles, which has been proposed in a scheme for transmitting signals through the upper ionosphere. This case is somewhat different from the one usually treated in that now not all portions of the wave-number surface can contribute to the radiation, but only those which intersect the plane representing the prescribed wave number of the moving current pattern on the antenna. We shall derive formal expressions for the far-field radiation pattern of such an antenna immersed in a uniform plasma and making an arbitrary angle with a uniform magnetic field. These expressions have been programmed for numerical evaluation on an on-line system, and plots of computed patterns for representative cases will be given in a later report.

We start from

$$\mathrm{curl}(\underset{\approx}{K}^{-1}\,\mathrm{curl}\,\underset{\sim}{H}) - k_o^2\,\underset{\sim}{H} = \mathrm{curl}(\underset{\approx}{K}^{-1}\,\underset{\sim}{J}) \qquad (1)$$

where $\underset{\approx}{K}$ is the dimensionless permittivity, which is assumed to have the form

$$\underset{\approx}{K} = \begin{pmatrix} \epsilon & ig & 0 \\ -ig & \epsilon & 0 \\ 0 & 0 & \eta \end{pmatrix} \tag{2}$$

with ϵ, η, g real as is the case for a cold plasma with the magnetostatic field in the z (3rd) direction. In these equations a time factor $\exp(-i\omega t)$ has been suppressed.

We choose the coordinate system so the antenna lies in the xz plane and makes an angle θ_o with the z axis. We also introduce a primed coordinate system with z' along the antenna and y' = y. We take the source term $\underset{\sim}{J}$ in Eq. (1), representing the current on the antenna, to have the form

$$\underset{\sim}{J} = I_o \, \hat{z}' \, (\pi/d^2) \, \exp(-\rho^2/d^2) \, \exp(ik_o n_o z') \tag{3}$$

Here I_o is the total current on the antenna, $\rho = (x'^2 + y'^2)^{1/2}$ is the distance from its axis, d is a scale length representing its thickness, n_o is the prescribed wave number along it and $\hat{z}'$ is a unit vector. In conjunction with the time factor $\exp(-i\omega t)$ this current distribution has the form of a traveling wave.

Taking the Fourier transform of (1) we find

$$(\underset{\sim}{k} \times \underset{\approx}{K}^{-1} \underset{\sim}{k} \times + k_o^2 \underset{\approx}{I}) \underset{\sim}{H}_k = -i \, \underset{\sim}{k} \times \underset{\approx}{K}^{-1} \underset{\sim}{J}_k \tag{4}$$

where $\underset{\sim}{H}_k$, $\underset{\sim}{J}_k$ are Fourier transforms and $\underset{\approx}{I}$ is the unit operator. We now put

$$\underset{\sim}{k} \times \underset{\approx}{K}^{-1} \underset{\sim}{k} \times = k_o^2 \sum_i \lambda_i \underset{\approx}{P}_i ; \quad \underset{\approx}{I} = \sum_i \underset{\approx}{P}_i \tag{5}$$

where λ_i are the eigenvalues and $\underset{\approx}{P}_i$ the corresponding projection operators of the symmetric operator $\underset{\sim}{k} \times \underset{\approx}{K}^{-1} \underset{\sim}{k} \times$, which are related to its normalized eigenvectors $\underset{\sim}{u}$ by $\underset{\approx}{P} = |u\rangle\langle u^\dagger|$. If we introduce a spherical coordinate system k, ξ, χ in wave-number space with the polar angle ξ measured from the magnetic field, then we can write the projection operators as

$$\underset{\approx}{P}_\pm = \frac{1}{2p_\pm} \begin{pmatrix} 0 & 0 & 0 \\ 0 & p_\pm + \alpha \sin^2\xi & -2i\,\chi\, g \cos\xi \\ 0 & 2\,i\,\chi\, g \cos\xi & p_\pm - \alpha \sin^2\xi \end{pmatrix} ; \quad \underset{\approx}{P}_o = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6}$$

where $\alpha = \epsilon(\epsilon - \eta) - g^2$ and $p_\pm = \pm(\alpha^2 \sin^4\xi + 4\chi^2 g^2 \cos^2\xi)^{1/2}$. The projection operator $\underset{\approx}{P}_o$ projects out the component along the wave-normal direction $\underset{\sim}{k}$, while $\underset{\approx}{P}_\pm$ project out the two left and right elliptically polarized components. The corresponding eigenvalues are

$$\lambda_\pm = - \frac{n^2}{2\eta(\epsilon^2 - g^2)} (2\,\epsilon\,\eta + \alpha \sin^2\xi - p_\pm) \tag{7}$$

$$\lambda_o = 0$$

where $n^2 = k^2/k_o^2$.

Substituting (5) into (4) we find

$$\sum_i (\lambda_i + 1)\, k_o^2\, \underset{\approx}{P}_i\, \underset{\sim}{H}_k = -i\underset{\sim}{k} \times \underset{\approx}{K}^{-1} \underset{\sim}{J}_k$$

Here i runs over the indices +, - and 0. We now multiply this equation on the left by $\underset{\approx}{P}_j$. Then, since $\underset{\approx}{P}_i^2 = \underset{\approx}{P}_i$, $\underset{\approx}{P}_{ij} = 0$, $i \neq j$, we find

$$\underset{\sim}{H}_k = \sum_i \underset{\approx}{P}_i \underset{\sim}{H}_k = \sum_i \frac{\underset{\approx}{P}_i \underset{\sim}{k} \times \underset{\approx}{K}^{-1} \underset{\sim}{J}_k}{i k_o^2 (\lambda_i + 1)} \tag{8}$$

This approach is equivalent, for example, to that of Deschamps and Kessler.[2)]

We now put

$$\begin{aligned} \underset{\approx}{P}_i \underset{\sim}{k} \times K^{-1} \hat{z} &= k_o \underset{\sim}{F}_i \\ \underset{\sim}{J}_k &= \hat{z} I_o Q \delta(k_\parallel - k_o n_o) \end{aligned} \tag{9}$$

Carrying out the computations using Eqs. (2), (3) and (6) we find

$$F_\pm = \frac{\eta \cos\xi}{p_\pm (p_\pm - a) n} \left\{ \left[(n^2 - \epsilon) \cos \chi + ig \sin \chi\right] \sin \theta_o \right.$$

$$\left. - \frac{n \sin\xi}{2 p_\pm \eta} \cos \theta_o \right\} \left[-ib \hat{\xi} + (p_\pm - a) \hat{\chi}\right] \tag{10}$$

$$Q = (4\pi^2)^{-1} \exp(- \tfrac{1}{4} k_\perp^2 d^2) \tag{11}$$

Here $a = \alpha \sin^2 \xi$, $b = 2\eta g \cos \xi$ and $k_\perp$, $k_\parallel$ refer to components perpendicular and parallel to the antenna, $k^2 = k_\perp^2 + k_\parallel^2$.

The solution of Eqs. (1) and (3) is given by the inverse Fourier transform of (8). After substituting (9), (10) and (11) into (8) and performing the integration over $k_\parallel$ we find

$$\underset{\sim}{H}(\underset{\sim}{r}) = (ik_o)^{-1} I_o \iint \sum_{\pm} \frac{\underset{\sim}{F}_\pm Q}{\lambda_\pm + 1} \exp(i\, \underset{\sim}{k} \cdot \underset{\sim}{r})\, d^2 \underset{\sim}{k} \tag{12}$$

where the integral is now over the surface $\underset{\sim}{k} \cdot \hat{z} = k_o n_o$. The summation is only over the two non-zero eigenvalues $\lambda_\pm$ since clearly $\underset{\sim}{F}_o = 0$. The far-field evaluation of (12) can now be carried out in two steps following a method of Lighthill.[3)] In the first the integral is evaluated as a sum of its residues on the two curves formed by the intersection of the plane $\underset{\sim}{k} \cdot \hat{z} = k_o n_o$ with the two-sheeted wave-number surface $\lambda_\pm = -1$. Thus (12) becomes the sum of two line integrals taken along these curves, which we call C_+ and C_-. In the second step these line integrals are evaluated by the method of stationary phase. If we introduce polar coordinates $k_\perp$, ψ for the wave number $\underset{\sim}{k}$ and ρ, φ for the field point $\underset{\sim}{r}$ in the plane $\underset{\sim}{k} \cdot \hat{z} = k_o n_o$, then we see that the phase $\underset{\sim}{k} \cdot \underset{\sim}{r} = k_+ \rho \cos(\varphi - \psi)$ is stationary at values of ψ satisfying

$$\frac{1}{k_\perp} \frac{d k_\perp}{d\psi} = \tan(\psi - \varphi) \tag{13}$$

These are the points on $C_\pm$ at which the normal to $C_\pm$ is in the direction of the field point, that is, in the direction φ. The result is

$$\underset{\sim}{H}(\underset{\sim}{r}) = I_o(ik_o)^{-1}\, 2\pi i\ \Sigma(2\pi/|\kappa|\,\rho)^{1/2}\ (\nabla\lambda)^{-1}\ Q\underset{\sim}{F}\ \exp(ik_\perp\,\rho\cos(\varphi-\psi) + \frac{1}{4}\, i\pi\,\mathrm{sgn}\,\kappa) \tag{14}$$

where κ is the curvature of the curve C_+ or C_- and $\nabla\lambda$ is the gradient of λ, both in the plane $\underset{\sim}{k}\cdot\hat{z} = k_o n_o$. The summation is now over all stationary phase points on both curves. All quantities appearing in the above expression, viz. κ, λ, Q, $\underset{\sim}{F}$, ψ are to be evaluated at each stationary phase point.

Since $\kappa^{-1} = ds/d\varphi$ where s is arc length along C, it is clear that

$$\kappa^{-1} = k_\perp\ (d\psi/d\varphi)\ \sec(\varphi-\psi) \tag{15}$$

Also

$$\begin{aligned}\nabla\lambda &= \frac{\partial\lambda}{\partial k_\perp}\ \sec(\varphi-\psi)\\ &= \frac{1}{k_o}\ \frac{k_\perp}{k}\ \frac{\partial\lambda}{\partial n}\ \sec(\varphi-\psi)\\ &= -\frac{2k_\perp}{k^2}\ \sec(\varphi-\psi)\end{aligned} \tag{16}$$

Here we have used (7) together with the fact that $\lambda_\pm = -1$ on $C_\pm$. Substituting (15) and (16) into (14) we obtain finally

$$\underset{\sim}{H}(\rho,\varphi) = \frac{1}{2}\ (2\pi)^{3/2} I_o \sum \frac{k^2}{k_o}\left[\frac{1}{k_\perp\rho}\ \frac{\partial\psi}{\partial\varphi}\ \cos(\varphi-\psi)\right]^{1/2} \underset{\sim}{F}\ Q$$

$$\times\ \exp\left[i\,k_\perp\,\rho\cos(\varphi-\psi) + \frac{1}{4}\ i\pi\,\mathrm{sgn}\,\kappa\right] \tag{17}$$

where the summation is over all points of $C_{\pm}$ satisfying (13). Note that sgn κ = sgn $\partial\psi/\partial\varphi = +1$ if the curve C is concave to the antenna and -1 otherwise.

It does not seem possible to express the quantities k, $k_{\perp}$, $\partial\psi/\partial\varphi$, ψ, $\underset{\sim}{F}$, Q appearing in (17) explicitly as functions of φ. Instead we express these quantities including φ itself as functions of ξ, the angle between $\underset{\sim}{k}$ and the magnetic field. From these expressions the required relations can be obtained numerically.

The x, y and z components of the vector relation $\underset{\sim}{k} = \underset{\sim}{k}_{\perp} + \underset{\sim}{k}_{\parallel}$, where $|k_{\parallel}| = k_o n_o$, are

$$k \sin \xi \cos \chi = k_{\perp} \cos \theta_o \cos \psi + k_{\parallel} \sin \theta_o$$

$$k \sin \xi \sin \chi = k_{\perp} \sin \psi \tag{18}$$

$$k \cos \xi = -k_{\perp} \sin \theta_o \cos \psi + k_{\parallel} \cos \theta_o$$

From these equations together with (13) we can find $k(\xi)$, $k_{\perp}(\xi)$, $\psi(\xi)$, $\chi(\xi)$, $\varphi(\xi)$ as explicit functions of ξ.

$$k^2(\xi) = k_o^2 \; \frac{2\eta\,(\epsilon^2 - g^2)}{2\epsilon\eta + \alpha \sin^2 \xi - p}$$

$$k_{\perp}(\xi) = (k(\xi)^2 - k_{\parallel}^2)^{1/2}$$

$$\psi(\xi) = \cos^{-1}\left[\frac{k_\parallel \cos\theta_o - k(\xi)\cos\xi}{k_\perp(\xi)\sin\theta_o}\right]$$

$$X(\xi) = \tan^{-1}\left[\frac{k_\perp(\xi)\sin\psi}{k_\parallel \sin\theta_o + k_\perp(\xi)\cos\theta_o\cos\psi}\right]$$

$$\varphi(\xi) = \psi(\xi) - \tan^{-1}\left(\frac{1}{k_\perp}\frac{dk_\perp/d\xi}{d\psi/d\xi}\right)$$

The symbol p appearing in the first equation above is defined after Eq. (6) and should bear the subscript ± referring to the curves $C_\pm$. Thus all the other quantities k, $k_\perp$, ψ, X, φ should bear this subscript whenever they appear in these equations, but for simplicity we have omitted it. By successive approximation these equations can be solved for $\xi(\varphi)$. Since $\xi(\varphi)$ is multiple-valued, we write $\xi_{i+}(\varphi)$, $\xi_{i-}(\varphi)$ to denote the set of values of ξ on C_+ and C_- for which the outward normal (in the $k_\parallel = k_o n_o$ plane) lies in the direction φ. Then the summation in (17) becomes a summation over these two sets of ξ. These equations have been programmed for numerical evaluation on an on-line system and plots of computed radiation patterns for representative cases will be included in a later report.

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